1 Recap

1. Examples of things that can go wrong in general.
2. Locally Lipschitz.
3. Uniqueness and uniformly locally Lipschitz.
4. Strategy for proving existence.

2 General existence and uniqueness theory

How can we hope to prove convergence? Well, we can try to look $y_{n+1} - y_n$ and if that decays quickly as $n \to \infty$, then y_n presumably converges. Thus, $y_{n+1}(t) - y_n(t) = \int_{t_0}^{t} (f(y_{n+1}(s), s) - f(y_n(s), s))ds$. This seems to be the right place to apply Gronwall and so on, provided we assume Lipschitzness.
Picard's theorem:

Theorem 1. *Let $D \subset \mathbb{R}^n$ be a domain and let $f : D \to \mathbb{R}^{n+1}$ be continuous (jointly) on D and Lipschitz in y on D with Lipschitz constant α. Let $R = \bar{B}_b(y_0) \times [t_0 - a, t_0 + a] \subset D$. Let $M = \max_R |f|$ and $h = \min\left(a, \frac{b}{M}, \frac{1}{2\alpha}\right)$. In the initial-value-problem (IVP) $y' = f(y,t)$ with $y(t_0) = y_0$ has a unique solution on $[t_0 - h, t_0 + h]$.*

Proof. Uniqueness was proven earlier. We shall prove existence by considering

$$y_{n+1}(t) = y_0 + \int_{t_0}^{t} f(y_n(s), s)ds \tag{1}$$

with $y_0(t) = y_0$. As mentioned earlier,

$$\|y_{n+1}(t) - y_n(t)\| \leq \alpha \int_{t_0}^{t} \|y_n - y_{n-1}\|ds, \tag{2}$$

provided $(y_n(s), s), (y_{n+1}(s), s) \in D$ for all s lying between t_0 and t. Suppose S_n is the set of all $t \in [t_0 - a, t_0 + a]$ such that $(y_n(t), t) \in \bar{B}_b(y_0) \times [t_0 - h, t_0 + h]$. Note that S_n is not empty. We claim that $[t_0 - h, t_0 + h] \subset S_n$. We shall prove this claim by induction on n. Clearly it is true for $n = 0$. Suppose it is true until $n - 1$. Then $y_n(t) - y(t_0) = \int_{t_0}^{t} f(y_{n-1}(s), s)ds$. Thus $\|y_n(t) - y_0\| \leq Mh \leq b$ if $t \in [t_0 - h, t_0 + h] \subset S_{n-1}$. Thus it is true for n.
As a consequence on $[t_0 - h, t_0 + h]$,

$$\begin{aligned} &\|y_{n+1} - y_n\| \leq \alpha \int_{t_0}^{t} \|y_n - y_{n-1}\|ds \\ &\Rightarrow \|y_{n+1} - y_n\|_{C^0[t_0-h,t_0+h]} \leq \alpha h \|y_n - y_{n-1}\|_{C^0[t_0-h,t_0+h]} \\ &\leq \frac{1}{2}\|y_n - y_{n-1}\|_{C^0[t_0-h,t_0+h]} \leq \frac{1}{2^n}\|y_1 - y_0\|_{C^0[t_0-h,t_0+h]}. \end{aligned} \tag{3}$$

Since $y_n = y_0 + (y_1 - y_0) + (y_2 - y_1) \ldots$, we see that by the Weierstrass M-test, this series converges uniformly to some limit y which is a continuous function. Since

f is continuous, $\int_{t_0}^t f(y_n(s), s)ds$ converges to $\int_{t_0}^t f(y(s), s)ds$ by uniform convergence (why?). Thus the limiting y satisfies $y = y_0 + \int_{t_0}^t f(y(s), s)ds$. By the fundamental theorem of calculus, y is differentiable and satisfies the IVP. $\square$