1 Recap

1. Picard's theorem (first version).

2 General existence and uniqueness theory

Theorem 1. *In the above theorem, h can be chosen to be* $\min\left(a, \frac{b}{M}\right)$.

Proof. Indeed, we claim that $\|y_n - y_{n-1}\| \leq \frac{M}{\alpha}\frac{(\alpha(t-t_0)))^n}{n!}$ for all $n \geq 1$. Indeed, for $n = 1$, it is easy. Assume inductively that it is true for $1, 2, \ldots, n$. Then

$$\begin{aligned}
\|y_{n+1} - y_n\| &\leq \alpha \int_{t_0}^{t} \|y_n - y_{n-1}\| ds \\
&\leq \alpha \int_{t_0}^{t} \frac{M}{\alpha}\frac{(\alpha(s-t_0)))^n}{n!} ds \\
&= \frac{M}{\alpha}\frac{(\alpha(t-t_0)))^{n+1}}{(n+1)!}. \qquad (1)
\end{aligned}$$

Since the series $\sum_n \frac{M}{\alpha}\frac{(\alpha h))^n}{n!}$ converges (why?), by the Weierstrass M-test, we are done as above (why?) $\square$

We now wish to characterise the maximal interval of existence. Here is a version of such a theorem (we are not stating it in the greatest generality possible but the technique of proof is generally applicable).

Theorem 2. *Let $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be locally Lipschitz. There exists a unique differentiable solution $y : (h_1, h_2) \to \mathbb{R}^n$ to $y' = f(y(t), t)$ with $y(t_0) = y_0$ where $h_1 < t_0 < h_2$ are extended real numbers such that (h_1, h_2) is the maximal interval of existence (what does this mean?). Moreover, if h_2 is finite, there exists a sequence $t_n \to h_2$ (with $t_n \in (h_1, h_2)$) such that $\|y(t_n)\| \to \infty$ and likewise for h_1.*

In other words, as long as y stays bounded, we can "continue" further. Equivalently, being unbounded is the only thing that can go wrong (this is in stark contrast to partial differential equations where higher order problems can play a role).

Proof. There surely is a solution on $[t_0 - h, t_0 + h]$ for some $h > 0$. Define h_2 as the supremum of all a_2 such that there is a solution on $[t_0 - h, a_2]$. By uniqueness there is a unique solution on $[t_0 - h, h_2)$. Likewise we can define h_1 and come up with the maximal interval of existence. Suppose h_2 is finite and $\|y(t)\| \leq C$ on $[t_0, h_2)$. (Why is this the negation of the hypothesis in the theorem?) Then since $y' = f(y(t), t)$, we see that $\|y'\| \leq C$ on $[t_0, h_2)$- why? (note that the constant C can vary from inequality to inequality) As a consequence, $\|y(s) - y(t)\| \leq |t - s|C$ and hence $y(h_2) := \lim_{t \to h_2^-} y(t)$ exists (why?). Because of uniform convergence, from $y(t) = y_0 + \int_{t_0}^{t} f(y, s)ds$ we see that $y(h_2) = y_0 + \int_{t_0}^{h_2} f(y, s)ds$. By the fundamental theorem of calculus, y is differentiable at h_2 and satisfies the ODE there. Thus we can extend the solution further using the existence theorem and arrive at a contradiction for maximality. $\square$

We can apply this result to prove that non-autonomous linear systems $y' = A(t)y + B(t)$ with $y(t_0) = y_0$ have unique solutions on $(-\infty, \infty)$. Indeed, $f(y,t) = A(t)y + B(t)$. Now $\|f(y_1,t) - f(y_2,t)\| \leq \|A(t)\|\|y_1 - y_2\|$ and hence it is locally Lipschitz. Thus if there is a solution, it is unique. Now by the existence theorem, the solution exists on some maximal interval (h_1, h_2). Note that if either of them (WLog h_2) is finite, then since $\|y(t)\| \leq C + \int_{t_0}^t (\|A(s)\|\|y(s)\| + \|B(s)\|)ds \leq C(1 + \int_{t_0}^t \|y(s)\|)$ on $[t_0, h_2)$, by Gronwall, y is bounded and hence by the previous theorem, we have a contradiction.
We can actually prove a more general existence theorem due to Peano.

Theorem 3. *Let f be continuous on a rectangle $R = \bar{B}_b(y_0) \times [t_0 - a, t_0 + a]$. Then there exists a solution (possibly non-unique) to $y' = f(y,t)$ with $y(t_0) = y_0$ on $[t_0 - h, t_0 + h]$ where $h = \min(a, b/M)$ where $M = \max_R |f|$.*

There are two proofs using approximations.

1. Using actual solutions to an approximation of the problem: Using the Stone-Weierstrass theorem (Let X be compact and Hausdorff. Let A be a subalgebra of $C(X, \mathbb{R})$ which contains a non-zero constant function. Then A is dense in $C(X, \mathbb{R})$ iff it separates points), we see that there is a sequence of smooth functions $f_n \to f$ on R uniformly.

2. Using approximate solutions to the original problem