1 Recap

1. Picard's theorem (second version).
2. Maximal interval of existence.
3. The Stone-Weierstrass theorem (recall) and its use to create an approximation to an IVP to prove Peano's theorem.

2 General existence and uniqueness theory

Theorem 1. *Let f be continuous on a rectangle $R = \bar{B}_b(y_0) \times [t_0 - a, t_0 + a]$. Then there exists a solution (possibly non-unique) to $y' = f(y,t)$ with $y(t_0) = y_0$ on $[t_0 - h, t_0 + h]$ where $h = \min(a, b/M)$ where $M = \max_R |f|$.*

There are two proofs using approximations.

1. Using actual solutions to an approximation of the problem: Using the Stone-Weierstrass theorem (Let X be compact and Hausdorff. Let A be a subalgebra of $C(X,\mathbb{R})$ which contains a non-zero constant function. Then A is dense in $C(X,\mathbb{R})$ iff it separates points), we see that there is a sequence of smooth functions $f_n \to f$ on R uniformly. (Another possibility is convolution with a nice function but that approximates only on a slightly smaller domain.) Now solve $y_n' = f_n(y_n, t)$ with $y_n(t_0) = y_0$ and $(y_n(t), t) \in \bar{B}_b(y_0) \times [t_0 - h_n, t_0 + h_n]$ where $h_n = \min(a, b/M_n)$ (this property follows from the proof). Note that $\|y_n'\| \leq M_n$. Thus $\|y_n(t) - y_n(s)\| \leq M_n\|t-s\|$ for all $t, s \in [t_0 - h_n, t_0 + h_n]$. Given $\epsilon > 0$, we can choose n large enough so that $|h_n - h| < \epsilon$. Thus for all such n, y_n is uniformly equicontinuous and uniformly bounded on $I_\epsilon = [t_0 - h + \epsilon, t_0 + h - \epsilon]$. Let ρ_ϵ be a smooth function with compact support in I_ϵ which is identity on $I_{2\epsilon}$. Thus $y_{n,\epsilon} = y_n \rho_\epsilon$ is defined on I_0 and is still uniformly equicontinuous and uniformly bounded on I_0. Choose a sequence $\epsilon_k \to 0$. The corresponding $y_{n_{\epsilon_k}, \epsilon_k}$ have a uniformly convergent subsequence (by Arzela-Ascoli) that converges uniformly to a continuous function y on I_0. This function is the desired solution (why?)

2. Using approximate solutions to the original problem: The idea is to use the Euler method to produce approximate solutions and hope that they converge (using Arzela-Ascoli again). Here is the precise definition of an ϵ-approximate solution: Let f be defined and continuous on a domain $D \subset \mathbb{R}^{n+1}$. An ϵ-approximate solution on $I = [t_0 - a, t_0 + a]$ is a function $y : I \to \mathbb{R}^n$ such that

   (a) $(t, y(t)) \in D$ for all $t \in I$.

   (b) y is C^1 on I except possibly for a finite set $S \subset I$ (but it has left and right derivatives on S).

   (c) $\|y' - f(y,t)\| \leq \epsilon$ on $I \cap S^c$.

   We produce ϵ-approximate solutions on $[t_0 - h, t_0 + h]$ for $h = \min(a, b/M)$ where f is continuous on $R = \bar{B}_b(y_0) \times [t_0 - a, t_0 + a]$ and $M = \max_R f$: We shall construct it on $[t_0, t_0+h]$ (and similar construction works on the other side). Divide the interval into subintervals and on each subinterval $[t_k, t_{k+1}]$ we give a linear approximation using the Euler method, i.e., Solve $z' = f(z_{k-1}, t_{k-1}), z(t_{k-1}) = z_{k-1}$ to get $z_k = z(t_k) = z_{k-1} + f(z_{k-1}, t_{k-1})(t_k - t_{k-1})$. But we need to choose the subintervals carefully so that these piecewise linear solutions all lie in R. Firstly, since f is uniformly continuous on R, $\|f(y,t) - f(\tilde{y}, \tilde{t})\| < \epsilon$ whenever $\|t - \tilde{t}\| + \|y - \tilde{y}\| \leq \delta$. We shall divide the interval into equal parts of size at most δ_1 which we shall choose later (it will turn out that $\delta_1 = \min(\delta, \delta/M)$ works). For any $t \in [t_{i-1}, t_i], \tilde{t} \in [t_{j-1}, t_j]$ (where $i \geq j$),

$$\begin{aligned} \|y(t) - y(\tilde{t})\| &\leq \|y(t) - y(t_j) + y(t_j) + \ldots + y(t_{i-1}) - y(\tilde{t})\| \\ &\leq M|t - t_j| + \ldots \leq M|t - \tilde{t}|. \end{aligned} \tag{1}$$

Hence if $\tilde{t} = t_0$, $\|y(t) - y(t_0)\| \leq Mh \leq b$. Thus we can easily show that by the choice of $\delta_1 = \min(\delta, \delta/M)$, y is an ϵ-approximate solution.

To be continued.....