1 Recap

1. First proof of Peano.
2. Construction of approximate solutions.

2 General existence and uniqueness theory

Theorem 1. *Let f be continuous on a rectangle $R = \bar{B}_b(y_0) \times [t_0 - a, t_0 + a]$. Then there exists a solution (possibly non-unique) to $y' = f(y,t)$ with $y(t_0) = y_0$ on $[t_0 - h, t_0 + h]$ where $h = \min(a, b/M)$ where $M = \max_R |f|$.*

We have produced approximate solutions. Now choose $\epsilon_n = \frac{1}{n}$. The corresponding approximate solutions y_n are uniformly bounded and uniformly equicontinuous (why?) and hence by Arzela-Ascoli, a subsequence (that we shall abuse notation and continue to denote as) y_n converges uniformly to a continuous function y. Uniform convergence now implies that y is a solution to the IVP. (Why?)

The next order of business to see how the solution depends on initial data as well as on other parameters that may occur in the differential equation (like physical constants for instance).

Theorem 2. *Let R be a "rectangle" as in Peano's theorem. Let $f, \tilde{f}$ be continuous on R, and uniformly Lipschitz in y with constants $\alpha, \tilde{\alpha}$. Let $y, \tilde{y}$ be the solutions of $y' = f(y,t), y(t_0) = y_0$, and $\tilde{y}' = \tilde{f}(\tilde{y}, t), \tilde{y}(\tilde{t}_0) = \tilde{y}_0$ in some closed interval I (containing $t_0, \tilde{t}_0$ and contained in R) of length $|I|$. Then*

$$\max_I \|y(t) - \tilde{y}(t)\| \leq e^{\min(\alpha,\tilde{\alpha})|I|} \left(\|y_0 - \tilde{y}_0\| + |I| \max_R \|f - \tilde{f}\| + \max(\|f\|_{\max_R}, \|\tilde{f}\|_{\max_R}) |t_0 - \tilde{t}_0| \right) \tag{1}$$

Proof.

$$\begin{aligned}
y - \tilde{y} &= y(t_0) - \tilde{y}(t_0) + \int_{t_0}^t (f(y(s),s) - \tilde{f}(\tilde{y}(s),s))ds \\
&= y(t_0) - \tilde{y}(\tilde{t}_0) + \tilde{y}(\tilde{t}_0) - \tilde{y}(t_0) + \int_{t_0}^t (f(y(s),s) - \tilde{f}(\tilde{y}(s),s))ds \\
\Rightarrow \|y - \tilde{y}\| &\leq \|y(t_0) - \tilde{y}(\tilde{t}_0)\| + \|\tilde{y}(\tilde{t}_0) - \tilde{y}(t_0)\| + \int_{t_0}^t \|f(y(s),s) - \tilde{f}(\tilde{y}(s),s)\|ds \\
&\leq \|y_0 - \tilde{y}_0\| + \max_I \|\tilde{y}'\||t_0 - \tilde{t}_0| + \int_{t_0}^t \|f(y(s),s) - \tilde{f}(\tilde{y}(s),s)\|ds \\
&\leq \|y_0 - \tilde{y}_0\| + \max_R \|\tilde{f}\||t_0 - \tilde{t}_0| + \int_{t_0}^t \|f(y(s),s) - \tilde{f}(y(s),s)\|ds + \int_{t_0}^t \|\tilde{f}(y(s),s) - \tilde{f}(\tilde{y}(s),s)\|ds \\
&\leq \|y_0 - \tilde{y}_0\| + \max_R \|\tilde{f}\||t_0 - \tilde{t}_0| + |I| \max_R \|f - \tilde{f}\| + \tilde{\alpha} \int_{t_0}^t \|y - \tilde{y}\|ds. \qquad (2)
\end{aligned}$$

Thus by Gronwall

$$\|y - \tilde{y}\| \leq e^{\tilde{\alpha}|I|}\left(\|y_0 - \tilde{y}_0\| + \max_R \|\tilde{f}\||t_0 - \tilde{t}_0| + |I| \max_R \|f - \tilde{f}\|\right) \tag{3}$$

Interchanging the roles of $y, \tilde{y}$, we are done. □

As a consequence, the solution depends continuously on the initial data and parameters involved. We can prove more. In fact, we can prove that if f is smooth, so is y. The rough idea is (for proving differentiability) to first pretend differentiability holds, deduce the ODE for the derivative, write an ODE for the difference quotient, subtract these two ODE and use the Gronwall inequality to deduce that indeed the difference quotient converges to the (hypothetical) derivative.