1 Recap

1. Second proof of Peano.
2. Continuous dependence on parameters.
3. Stated the C^k dependence theorem and reduced it to $k = 1$.

2 General existence and uniqueness theory

We shall now prove that if f and $y_0 = y(0)$ are C^1 functions of $v \in \mathbb{R}^m$ then so is y. All we need to do is to prove that $\frac{\partial y}{\partial v_i}$ exists and is continuous for all i. Fix i and call v_i as w. We need to identify the (hypothetical) derivative y_w and prove that indeed it is the correct derivative. To this end, consider the following ODE obtained by differentiating the original IVP. (Later we will show that $u = y_w$.) Let's work with one function y (as opposed to a vector) for simplicity. The proof doesn't change much otherwise.

$$\begin{aligned} u' &= f_y u + f_w \\ u(0) &= (y_0)_w. \end{aligned} \tag{1}$$

Now this ODE is a linear system for u. So it has a unique differentiable (in t) solution for as long as the right-hand-side makes sense. Moreover, the right-hand-side is continuous and hence u is continuous jointly in t, v.
Consider the equation satisfied by the difference-quotient $\Delta_h y = \frac{y(t,v_1,\ldots,w+h,\ldots)-y(t,v_1,\ldots,w,\ldots)}{h}$.

$$\begin{aligned} (\Delta_h y)' &= \frac{f(y(t,\ldots,w+h,\ldots),t,\ldots,w+h,\ldots) - f(y(t,\ldots,w,\ldots),t,\ldots,w,\ldots)}{h} \\ \Delta_h y &= \Delta_h(y_0). \end{aligned} \tag{2}$$

Now subtract to get

$$(\Delta_h y - u)' = \frac{f(y(t,v) + \Delta_h y, t, \ldots, w+h, \ldots) - f(y(t,v),t,v)}{h} - f_y u - f_w \Delta_h y - u = \Delta_h(y_0) - (y_0)_w \tag{3}$$

Thus

$$\begin{aligned} \|\Delta_h y - u\| &\leq \|\Delta_h(y_0) - (y_0)_w\| \\ &+ \int_0^t \left\| \frac{f(y(s,v) + \Delta_h y, s, \ldots, w+h, \ldots) - f(y(s,v),s,v)}{h} - f_y u - f_w \right\| ds \\ &\leq \|\Delta_h(y_0) - (y_0)_w\| + \int_{t_0}^t \Big(\|f_y(\xi) - f_y\| \|\Delta_h y\| \\ &+ C\|\Delta_h y - u\| + \|f_w(\xi) - f_w\| \Big) ds \\ &\leq \epsilon + C \int_{t_0}^t \|\Delta_h y - u\| ds, \end{aligned} \tag{4}$$

for all $0 < \|h\| < \delta$ (depending on ϵ). By Gronwall, we can see that indeed y is partially differentiable and u is its derivative w.r.t w (why?) Now since u is continuous, y is C^1. $\square$

3 A little bit about numerical methods

It is crucial to find efficient algorithms to solve ODE on computers. (After all, we know most of these explicit methods are slow if nothing else.) We want to solve $y' = f(y,t)$ (where y, f are vector-valued) with $y(t_0) = y_0$. By "solve", we want $y(t_1)$ for any given t_1 quickly (Wlog $t_1 > t_0$) and approximately to a given error ϵ. We are "given" $f(y,t)$ in that we can assume that there is a subroutine that provides $f(y,t)$ quickly and exactly to us. (Actually, it cannot be given exactly but it is easiest to assume this hypothesis.) We assume that $f(y,t)$ is smooth (for simplicity) on $\mathbb{R}^{n+1}$. Thus there exists a unique solution for some time (p,q). We assume that t_1 lies in this interval. We also assume that on $[t_0,t_1]$, $\|y''\| \leq M$, and that f is actually Lipschitz on $\mathbb{R}^{n+2}$ in y with constant L.

The simplest algorithm is due to Euler: Divide $[t_0,t_1]$ into equal pieces of size h. Then define $y_n = y_{n-1} + f(y_{n-1},t_{n-1})h$. We hope that if h is sufficiently small, then $\|y_N - y(t_1)\| < \epsilon$ where y is the actual solution and $N = \frac{t_1-t_0}{h}$. Now $y(t_n) = y(t_{n-1}) + \int_{t_{n-1}}^{t_n} f(y(s),s)ds$. Now $\|y(t_n) - y(t_{n-1}) - hy'(t_{n-1})\| \leq Mh^2$ (Taylor's theorem). Thus $\|y(t_n) - y(t_{n-1}) - hf(y(t_{n-1}),t_{n-1})\| \leq Mh^2$. Now

$$\begin{aligned}
\|y(t_n) - y_n\| &\leq Mh^2 + \|y_{n-1} + hf(y_{n-1},t_{n-1}) - y(t_{n-1}) - hf(y(t_{n-1},t_{n-1}))\| \\
&\leq \|y_{n-1} - y(t_{n-1})\|(1+Lh) + Mh^2 \\
&\Rightarrow \|y(t_n) - y_n\| \leq \frac{M((1+Lh)^n - 1)}{L}h \\
&\Rightarrow \|y_N - y(t_1)\| \leq \frac{Mh}{L}(e^{(t_1-t_0)L} - 1). \qquad (5)
\end{aligned}$$

The bottom line is that the error is linear in h. (Hence Euler is sometimes called a first-order method.) The problem with Euler's method is not just that it is slow. It is unstable. For instance, if $y' = -ky$, then y decays exponentially. However, depending on the step size h, the numerical solution can oscillate and do other crazy things. Do note that rounding-off errors play a big role too. For instance if h is too small, then while truncation errors are small, the machine rounding off errors can be large. (We add lots of small numbers.) So one needs to resort to complicated summation approaches too. (Like compensated summation.) The other thing is that f itself may not be specified exactly and that also introduces errors.