1 Recap

1. Smooth dependence on parameters theorem.

2. Euler's method.

2 A little bit about numerical methods

A better method (at least as far as truncation error is concerned) is the midpoint method: It uses the observation that $y'(t+h/2) \approx \frac{y(t+h)-y(t)}{h}$. Moreover, $y(t+h/2) \approx y(t) + \frac{h}{2}y'(t)$. Thus, $y_{n+1} = y_n + hf(t_n + h/2, y_n + \frac{h}{2}f(y_n, t_n))$. The global truncation error is roughly $O(h^2)$. Indeed, $y'(t+h/2) - \frac{y(t+h)-y(t)}{h} = O(h^2)$ and hence in each step the error is $O(h^3)$. So the global error is roughly $O(h^2)$. This leads to the (explicit) Runge-Kutta methods: $y_{n+1} = y_n + h\sum b_i k_i$ where $k_1 = f(t_n, y_n)$, $k_2 = f(t_n + c_2 h, y_n + ha_{21}k_1)$, $k_3 = f(t_n + c_3 h, y_n + h(a_{31}k_1 + a_{32}k_2))$, and so on. (There are implicit methods as well.)
A rough idea behind why the midpoint method works: $\int_t^{t+h} y'(s)ds - y'(t+h/2)h = \int_t^{t+h}(y'(t) + y''(t)(s-t) + O((s-t)^2)ds - y'(t) - y''(t)h^2/2 + O(h^3) = O(h^3)$.

3 Sturm-Liouville theory

Consider a vibrating elastic rod of density $\rho(x)$ and tension $k(x)$. Then suppose we take an infinitesimal element dx between $x, x+dx$. Suppose it moves up by $y(x)$. Then the net vertical force on it is $k(x+dx)\sin(\theta(x+dx)) - k(x)\sin(\theta(x)) = dk\sin(\theta(x)) + k(x)\cos(\theta(x))dx = \rho(x)dx\frac{\partial^2 y}{\partial t^2}$. Thus

$$\frac{\partial}{\partial x}\left(k(x)\sin(\theta(x))\right) = \rho(x)\frac{\partial^2 y}{\partial t^2} \tag{1}$$

For small θ, $\sin(\theta) \approx \frac{\partial y}{\partial x}$. Thus we get the wave equation

$$\frac{\partial}{\partial x}\left(k(x)\frac{\partial y}{\partial x}\right) = \rho(x)\frac{\partial^2 y}{\partial t^2}. \tag{2}$$

Now we substitute $y = u(x)\cos(\nu t)$ to get

$$(ku')' = -\rho\nu^2 u(x). \tag{3}$$

For a finite rod $x \in [a, b]$ here are some natural boundary conditions:
$u(a) = u(b) = 0$ (rigid ends), $u'(a) + \alpha u(a) = u'(b) + \beta u(b) = 0$ (elastically held ends), $u(a) = u(b), u'(a) = u'(b)$ (periodic boundary conditions).
For example if k, ρ are constants (set to 1 by choosing units appropriately), and $a = 0, b = 2\pi$, then $u'' = -\nu^2 u$ with $u(0) = u(\pi) = 0$. Thus $u = A\sin(\nu x)$ where ν is an integer. Thus the "eigenvalues" of the corresponding linear map from the vector space of smooth functions to itself are discrete and have no limit points. Moreover, it turns out that any "reasonable function" can be "written using sines and cosines". This theorem of Fourier series needs Lebesgue integration and a little bit of functional

analysis to state and prove.
Motivated by this example we study the general *regular/elliptic* Sturm-Liouville problem:

$$Lu = -(pu')' + qu = \lambda \rho u, \tag{4}$$

where $p > 0, \rho > 0$, p is in C^1, q, ρ are continuous and $\lambda \in \mathbb{R}$ on $[a, b]$. The boundary conditions will be of the form

$$\begin{aligned} \alpha_1 u(a) + \alpha_2 u'(a) = \alpha_3,\ |\alpha_1| + |\alpha_2| > 0 \\ \beta_1 u(b) + \beta_2 u'(b) = \beta_3,\ |\beta_1| + |\beta_2| > 0. \end{aligned} \tag{5}$$