1 Recap

1. Midpoint method.
2. Sturm-Liouville boundary value problem.

2 Sturm-Liouville theory

We now have the following theorem (akin to its finite-dimensional counterpart).

Theorem 1. *Consider the BVPs:*

$$Lu = r, \tag{1}$$

where r is continuous on $[a, b]$, and

$$Lu = 0,\ \alpha_3 = 0 = \beta_3. \tag{2}$$

Then the following hold (Fredholm's alternatives).

1. *If 2 has only a trivial solution, then 1 has a unique solution.*
2. *If 2 has a non-trivial solution, then 1 has infinitely many solutions, if it has one solution.*

Proof. 1. If u_1, u_2 are two solutions of 1, then $u_1 - u_2$ is a solution of 2. Hence if there is a solution, it is unique. Now we prove existence. We already know that $Lu = 0$ has two linearly independent solutions u_1, u_2 by the existence theorem for linear non-autonomous systems. Moreover, there exists at least one solution u_0 to $Lu = r$ (with in fact given initial data. Let's say with $u(a) = 0 = u'(a)$). We aim to find a solution to 1 of the form $u_0 + c_1u_1 + c_2u_2$. We simply need to choose c_1, c_2 so that the boundary conditions are met. This will happen if the matrix $A = \begin{bmatrix} \alpha_1 u_1(a) + \alpha_2 u_1'(a) & \alpha_1 u_2(a) + \alpha_2 u_2'(a) \\ \alpha_1 u_1(b) + \alpha_2 u_1'(b) & \alpha_1 u_2(b) + \alpha_2 u_2'(b) \end{bmatrix}$ is invertible. If A is not invertible, then its kernel will produce c_1, c_2 such that $c_1u_1 + c_2u_2$ is a non-trivial solution of 2.

2. Indeed, if u_0 is a solution and u is a non-trivial solution of the homogeneous problem, then $u_0 + cu$ is a solution for all $c \in \mathbb{R}$.

□

We now prove some properties about the linear map L.

Theorem 2. 1. *L is symmetric, i.e., $\int_a^b Luv = \int_a^b uLv$ (for u, v satisfying the boundary conditions).*

2. *Suppose u, v are C^2 functions that satisfy the "eigenvalue" equations $Lu = \lambda\rho u$, $Lv = \mu\rho v$ where $\lambda \neq \mu$ (and the boundary conditions), then u, v are "orthogonal" in the sense that $\int_a^b \rho uv = 0$.*

Proof. The key point is to prove Lagrange's identity: For any two C^2 functions u, v,

$$vLu - uLv = (pW)', \tag{3}$$

where $W = uv' - u'v$ (why?) Thus upon integration, we see that L is symmetric. Now $0 = \lambda \int_a^b \rho uv - \mu \int_a^b \rho uv$ and hence u, v are orthogonal. □

One of our main aims is to prove the existence of eigenfunctions.

Theorem 3. *The eigenvalues of the regular Sturm-Liouville problem (with $\alpha_3 = \beta_3 = 0$) is an infinite sequence $\lambda_0 < \lambda_1 \ldots$ converging to ∞. The eigenfunction u_n corresponding to λ_n is unique upto a constant factor and has exactly n zeroes in (a, b).*

We can of course find a solution on $[a, b]$ satisfying the boundary condition at a. The challenge is to make sure that the boundary condition at b is also met. The idea will be to take all possible initial conditions at a (satisfying the boundary condition at a), and seeing how this 1-parameter family of u behaves at b. We shall show that the boundary values vary between all possible values (via the intermediate value theorem). We also need to study the "oscillation" of u to know how many zeroes exist. Uniqueness is actually straightforward. Suppose u, v are eigenvectors with the same eigenvalue. Then the boundary condition at a implies that the Wronskian at a is 0 and hence is zero throughout. Thus they are linearly dependent (from a previous HW problem).