1 Recap

1. Proved that any solution to SL has only finitely many zeroes.
2. Defined the Prüfer substitution (using path lifting) and derived ODE for θ, r. The ODE for r was incorrect. The point however is that the boundary values for SL correspond to boundary values for θ. Proved that there exists a unique solution for some short period of time.

2 Sturm-Liouville theory

$$\theta' = (\lambda\rho - q)\sin^2\theta + \frac{1}{p}\cos^2(\theta) = F(t,\theta),$$
$$r' = \frac{1}{2}\left(\frac{1}{p} - (\lambda\rho - q)\right) r\sin(2\theta). \tag{1}$$

$$r = r(a)\exp\left(\frac{1}{2}\int_a^t \left(\frac{1}{p(s)} - (\lambda\rho - q)(s)\right)\sin(2\theta(s))ds\right).$$

Note that changing $r(a)$ only scales u by a constant factor. Hence the zeroes of u can be located by studying θ.
Since the right-hand-side is bounded, θ' is bounded and hence θ stays bounded in the maximal interval. Thus the maximal interval is $[a, b]$.
Zeroethly, note the following observations.

1. If θ is a solution, then so is $-\theta$ (with the same eigenvalue). Hence, $\theta(a)$ can be assumed to be ≥ 0. Moreover, by subtracting enough multiples of π, $\theta(a) \in [0, \pi)$.
2. To find zeroes, simply find those values of t for which $\theta(t) = n\pi$.
3. Let $n \geq 0$. If there is a t_n so that $\theta(t_n) = n\pi$, then $\theta'(t_n) > 0$ and hence $\theta(t) > n\pi$ for $t > t_n$ and sufficiently close to t_n. In fact, if there is a $t > t_n$ such that $\theta(t) = n\pi$, then θ' at that point is ≤ 0 which is a contradiction. Thus, $\theta > 0$ on $[a, b]$.

The point is the following oscillation theorem.

Theorem 1. *Let $\theta(t, \lambda)$ be a solution of the above ODE with $\theta(a, \lambda) = \gamma \in [0, \pi)$. Then θ is continuous and it is strictly increasing in λ. Moreover, $\lim_{\lambda\to\infty}\theta(t, \lambda) = \infty$ and $\lim_{\lambda\to-\infty}\theta(t, \lambda) = 0$ for any $t \in (a, b]$.*

Given this theorem, let us prove the main theorem of SL: The boundary condition on b can be stated as $\theta(b, \lambda) = \delta + n\pi$ for $n = 0, 1, \ldots$ provided δ satisfies $\beta_1\sin(\delta) + \beta_2\frac{\cos(\delta)}{p(b)} = 0$. There is of course a unique $\delta \in (0, \pi]$ satisfying it. For this value of δ, by the theorem above and the intermediate value theorem, there is a unique λ such that $\delta = \theta(b, \lambda)$. Call this λ_0. (Note that u_0 does not vanish on (a, b).) Likewise, we can produce $\lambda_1, \ldots$ which form an increasing sequence. Since $\delta_n \to \infty$, so does λ_n. Why does the statement about zeroes follow? □

Now we shall prove the oscillation theorem. We first prove that θ is strictly increasing in λ. Note that $\theta' = (\lambda\rho - q)\sin^2(\theta) + \frac{\cos^2(\theta)}{p}$. Now since the right-hand-side is C^1 in λ, $\theta(t, \lambda)$ is differentiable with respect to λ and the derivative is (jointly) continuous (and $\theta(t, \lambda)$ is continuous jointly). We differentiate the equation.

$$\begin{aligned}\left(\frac{\partial\theta}{\partial\lambda}\right)' &= \rho\sin^2(\theta) + (\lambda\rho - q)\sin(2\theta)\frac{\partial\theta}{\partial\lambda} - \frac{\sin(2\theta)}{p}\frac{\partial\theta}{\partial\lambda} \\ &= \rho\sin^2(\theta) + \sin(2\theta)\frac{\partial\theta}{\partial\lambda}\left(\lambda\rho - q - \frac{1}{p}\right) \\ &\qquad \frac{\partial\theta}{\partial\lambda}(a, \lambda) = 0. \end{aligned} \tag{2}$$

Thus using an integrating factor, we can solve for $\frac{\partial\theta}{\partial\lambda}$ and see that it is > 0. Hence θ is strictly increasing in λ.

Now note that for λ large enough, $\theta' \geq \frac{\lambda\sin^2(\theta)+1}{C}$ (why?) This means that θ is strictly increasing. If $\lim_{\lambda\to\infty}\theta(t, \lambda) = L < \infty$ for some $t = t_0$, then the limit is $\leq L$ for all $t \in (a, t_0]$. Thus there are $N + 1$ zeroes of $\sin^2(\theta)$ in $[a, t_0]$. Therefore, for at least an interval $[v, w]$ of size $\frac{t_0-a}{N+3}$, we know that $k\pi \leq \theta \leq (k + 1)\pi$. On this interval of t, $\int_{k\pi}^{(k+1)\pi} \frac{d\theta}{\lambda\sin^2(\theta)+1} \geq \frac{w-v}{C}$. Upon explicit integration (by taking $t = \tan(\theta)$ for instance), we arrive at a contradiction.