1 Recap

1. Proved that there is a solution to the equation for θ on $[a, b]$.

2. Also proved that if the oscillation theorem

   Theorem 1. *Let $\theta(t, \lambda)$ be a solution of the above ODE with $\theta(a, \lambda) = \gamma \in [0, \pi)$. Then θ is continuous and it is strictly increasing in λ. Moreover, $\lim_{\lambda\to\infty} \theta(t, \lambda) = \infty$ and $\lim_{\lambda\to-\infty} \theta(t, \lambda) = 0$ for any $t \in (a, b]$.*

   is true, then the SL theorem can be proven.

3. Proved most of the oscillation theorem (except for the second limit).

2 Sturm-Liouville theory

A small digression: We already know that regardless of λ, if $\theta(t_n) = n\pi$, then for all $t > t_n$, $\theta(t) > n\pi$. This means that the nth zero of u occurs when $t = n\pi$. Note that when $\lambda \to \infty$, if there is a sequence λ_k such $\lim_k t_n(\lambda_k) > a$, then again the above argument applies to arrive at a contradiction. This means that $t_n(\lambda) \to a$ when $\lambda \to \infty$.

Now we prove the second limit. We already know that $\theta > 0$ on $(a, b]$. Fix $t = t_1$. Suppose given an $\epsilon > 0$, we produce a λ such that $\theta(t_1, \lambda) < \epsilon$, we are done. To this end, (by shrinking ϵ if necessary), assume that there is a $\epsilon < \gamma < \gamma_1 \leq \pi - \epsilon$. Now consider a straight line $s(t)$ joining (a, γ_1) and (t_1, ϵ) (with negative slope m). Note that if for some λ, the graph of $\theta(t, \lambda)$ lies below $s(t)$, then $\theta(t_1, \lambda) < s(t_1) = \epsilon$ and we are done. It is easy to see that θ lies below the straight line for some $[a, a_1]$. Suppose for the sake of contradiction that there is a t so that $\theta(t, \lambda) > s(t)$. Then choosing the smallest such $t = t_*$, $\theta(t_*, \lambda) = s(t_*)$ and $\theta'(t_*, \lambda) \geq m$. Since $\theta(t_*) = s(t_*) = \gamma_1 + m(t_* - a)$, substituting for m and the upper bound for γ_1 we see that $\theta(t_*, \lambda) \in [\epsilon, \pi - \epsilon]$ and hence $\sin(\theta_*) \geq \sin(\epsilon)$. Thus for sufficiently negative λ, $\theta'(t_*, \lambda) < m$ and that is a contradiction. ☐

We can in fact produce a good estimate for where the zeroes of u lie. To prove such an estimate, one proves a very important result (which is useful in Riemannian geometry too) called the Sturm comparison theorem for ODE of the form $(P_i u_i)' + Q_i u_i = 0$ for $i = 1, 2$ where $P_i > 0$ are C^1, and Q_i are continuous. If θ_i are the corresponding Prüfer phases, then $\theta_i' = Q_i \sin^2(\theta_i) + \frac{1}{P_i} \cos^2(\theta_i) = F_i(t, \theta_i)$.

Theorem 2. *Assume $P_1 \geq P_2 > 0$ and $Q_1 \leq Q_2$. Then between any two zeroes of a non-trivial u_1, there is at least one zero of every u_2 except if $u_2 = cu_1$. In the latter case, we have $P_1 = P_2, Q_1 = Q_2$ everywhere except possibly on the set $Q_1 = Q_2 = 0$.*

Proof. Suppose we have two zeroes $\theta_1(a) = n\pi$ and $\theta_1(b) = (n + 1)\pi$ of u_1. Choose the initial data for θ_2 to be such that $(n + 1)\pi > \theta_2(a) \geq \theta_1(a) = n\pi$. We claim that

$$\theta_1(t) \leq \theta_2(t) \;\forall\, t \in [a, b],$$

and that $\theta_1(b) = \theta_2(b)$ iff $\theta_1 \equiv \theta_2$.

Assume this claim. Then we see that $\theta_1(b) \leq \theta_2(b)$ and therefore unless $\theta_2(b) = (n+1)\pi$, θ_2 will cross $(n+1)\pi$ before b and hence u_2 will have a zero. If $\theta_2(b) = \theta_1(b)$, then $\theta_1(t) = \theta_2(t) = \theta$. Subtracting the two equations,

$$(Q_2 - Q_1)\sin^2(\theta) + \left(\frac{1}{P_2} - \frac{1}{P_1}\right)\cos^2(\theta) = 0.$$

When $\sin(\theta) = 0$, $\theta' > 0$ and hence the zeroes of u_i are isolated. Thus $Q_2 = Q_1$ everywhere and $P_2 = P_1$ unless $\cos(\theta) = 0$. On such intervals where $\cos(\theta) = 0$, θ is constant. Thus on such intervals, $Q_1 = Q_2 = 0$ because $\sin^2(\theta) = 1$.
To be continued.... □