1 Recap

1. Finished the proof of the oscillation theorem.

2. Stated the Sturm comparison theorem for ODE of the type $(P_i u_i')' + Q_i u_i = 0$ where $P_1 \geq P_2 > 0$, $Q_1 \leq Q_2$. We proved it except for the case where $\theta_1(t) \equiv \theta_2(t) = \theta$.

2 Sturm-Liouville theory

Proof. Continued....
On the intervals where $\cos(\theta)$ does not vanish, $P_1 = P_2$ and therefore, both u_1, u_2 satisfy the same second order linear homogeneous ODE. The Wronskian vanishes. Thus $u_2 = cu_1$ on each such interval. Potentially the c can be different from interval to interval. Now consider some open interval where $u_2 = cu_1$ and let $S \subset [a, b]$ be the set of points where $u_2 = cu_1$. S is non-empty and closed. It is open too: Suppose $u_2(t) = cu_1(t)$. If $\cos(\theta(t)) \neq 0$, then indeed $u_2 = cu_1$ for a neighbourhood of t. If $\cos(\theta(t)) = 0$, then either there is a sequence of t_n such that $\cos(\theta(t_n)) \neq 0$ converging to t in which case t_n is in the boundary of one of the intervals on which $\cos(\theta) \neq 0$ and hence by continuity, on that interval $u_2 = cu_1$. On the other side of t, either the same happens or $\cos(\theta) \equiv 0$ on the other side in which case $\theta' = 0$ and hence $Q_2 = Q_1 = 0$ which means that u_1, u_2 are constants and thus $u_2 = cu_1$ by continuity. □

We just have to prove the claim. To this end, we see that

$$(\theta_1 - \theta_2)' = Q_2(\sin^2(\theta_1) - \sin^2(\theta_2)) + \frac{1}{P_2}(\cos^2(\theta_1) - \cos^2(\theta_2))$$
$$(\theta_1 - \theta_2)(a) \leq 0. \tag{1}$$

Now suppose the claim is false. Then (why?) there exists an interval $[a_1, b_1]$ such that $(\theta_1 - \theta_2)(a_1) = 0$ and $(\theta_1 - \theta_2)(t) > 0$ on $[a_1, b_1]$. In that case, on this interval (why?),

$$(\theta_1 - \theta_2)' \leq C|\theta_1 - \theta_2| = C(\theta_1 - \theta_2)$$
$$(\theta_1 - \theta_2)(a_1) = 0 \tag{2}$$

Thus by Gronwall, $\theta_1 \leq \theta_2$ which is a contradiction. Therefore, $\theta_1 \leq \theta_2$ on $[a, b]$. In fact, if $\theta_2(b) = \theta_1(b)$, then suppose $\theta_1(t) < \theta_2$ on a subinterval. Then the above argument shows that $\theta_1(b) < \theta_2(b)$ which is a contradiction and hence if equality holds, $\theta_1 \equiv \theta_2$. □

Now we prove a theorem about the location of the zeroes. (The proof will be done assuming $\alpha_1 \neq 0$. A similar proof (for instance, changing t to $a + b - t$ when $\beta_1 \neq 0$, and taking $\gamma = \frac{\pi}{2}$ if both are zero) works otherwise.

Theorem 1. *Let q_M, p_M, ρ_M be the maxima of q, p, ρ over $[a, b]$ and q_m, p_m, ρ_m be the minima. Let u be a non-trivial solution (with if $\lambda > 0, \lambda\rho_m - q_M > 0, \lambda\rho_M - q_m > 0$) of the Sturm-Liouville problem with $\tan(\gamma) = \frac{u(a)}{p(a)u'(a)}$. If t_n denotes the nth zero of u in (a, b), then*

$$\frac{\sqrt{p_m}}{\sqrt{\lambda\rho_M - q_m}} \leq \frac{t_n - a}{n\pi - \gamma} \leq \frac{\sqrt{p_M}}{\sqrt{\lambda\rho_m - q_M}}. \tag{3}$$

Proof. Consider the equations

$$p_M \nu'' + (\lambda \rho_m - q_M)\nu = 0, \tag{4}$$

and

$$p_m \mu'' + (\lambda \rho_M - q_m)\mu = 0. \tag{5}$$

We can find explicit sinusoidial solutions of these equations with $\tan(\gamma) = -\frac{\alpha_2}{\alpha_1}$ for both initially. By the Sturm comparison theorem, any solution u will have a zero between two zeroes of ν. Likewise, the zeroes of μ will lie between those of u. Thus the theorem follows. $\square$