1 Recap

1. Finished the proof of the Sturm comparison theorem for ODE of the type $(P_i u_i')' + Q_i u_i = 0$ where $P_1 \geq P_2 > 0$, $Q_1 \leq Q_2$.

2. Applied it to zeroes.

2 Stability - Linear theory

We want to study $\vec{x}' = A\vec{x}$ where A is a given $n \times n$ real matrix. We of course know the solution: $\vec{x} = e^{At}\vec{x}_0$. The point is to see what happens as $t \to \infty$. Note that if $\vec{x}_0$ is such that $A\vec{x}_0 = 0$, then $\vec{x} \equiv \vec{x}_0$ for all time! Such a vector is called an equilibrium point or a steady state solution.
For instance, if we take $x'' = -k^2 x$, then $x = 0$ is the only equilibrium point (corresponding to the mean position of the pendulum/spring). In this example, if we perturb a little bit from equilibrium, we will oscillate. Indeed, if we model it as

$$\begin{bmatrix} v' \\ x' \end{bmatrix} = \begin{bmatrix} 0 & -k^2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v \\ x \end{bmatrix},$$

then we note that the eigenvalues of A are $\pm\sqrt{-1}k$. Thus A is diagonalisable and $\|e^{At}\| \leq C$ for all t (why?). Thus we will never move very far from equilibrium but it isn't like we will eventually return to equilibrium either. On the other hand, if we add some damping, $x'' = -k^x - bv$, then the eigenvalues of A have a negative real part and hence by one of the HW problems, $\|e^{At}\|$ decreases exponentially. In this case, we will always return to the mean position.

2.1 2×2 matrices

In other words, the eigenvalues of A play a crucial role in deciding what happens to this "dynamical system" eventually.
Now consider another example: $x' = -x, y' = 2y$. This is a decoupled system whose solution is $x = x_0 e^{-t}$, $y = y_0 e^{2t}$. That is, if we fix t, $y = \frac{A}{x^2}$. In other words, the "particle" moves on one of these curves. Solutions to these sorts of equations are called trajectories and the collection of all trajectories is called a phase portrait. We can plot these on the $x - y$ plane to get a better idea. In fact, we can do more picture stuff. If we consider the vector field $(-x, 2y)$, and plot it, note that by Euler's method, we can roughly see where a particle will move given its initial data. (By the way, if we pretend that this vector field is the velocity field of water, and the particle is a tiny paper boat, the map $(x_0, y_0) \to (x(t), y(t))$ is called the time-t flow of the vector field.) This way, graphically, we can get an idea without solving the equation. Note that the equilibrium point at the origin is somewhat strange in that if we perturb a bit towards any direction other than the x-direction, the particle will run off to infinity. Such an equilibrium is called a *saddle point* and it is unstable.
Now consider $A = diag(\lambda, \lambda)$. In this case, if $\lambda > 0$, then the equilibrium is unstable. If

$\lambda < 0$ it is stable. In both cases, it is called a *node*. The same is the case with $A = \lambda I + N$ (Jordan canonical form).
Now consider $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. The solution is $e^{at} \begin{bmatrix} \cos(bt) & -sin(bt) \\ \sin(bt) & \cos(bt) \end{bmatrix} \vec{x}_0$. We have a few cases to consider:

1. $a = 0$: In this case, $\|x(t)\| = \|x_0\| \; \forall \, t$. This system is "oscillating" in a sense. The equilibrium at the origin is called a *centre*.

2. $a \neq 0$: In this case the equilibrium is called a focus. If $a > 0$, it is an unstable focus and if $a < 0$ it is a stable focus.

We now make the definitions more rigorously. Let A be a 2×2 matrix whose Jordan canonical form is B.

1. If $B = diag(\lambda, \mu)$ is real where $\lambda\mu < 0$, zero is called a saddle point.

2. If $B = diag(\lambda, \mu)$ is real with $\lambda\mu > 0$ or $B = \lambda I + N$ where $\lambda \neq 0$, then 0 is called a node.

3. If $B = diag(\lambda, \mu)$ is complex with the eigenvalues being $a \pm ib$ where $a \neq 0, b \neq 0$, then 0 is called a focus.

4. If $B = diag(\lambda, \mu)$ is complex with the eigenvalues being $\pm ib$ where $b \neq 0$, then 0 is called a centre.

A stable node or focus is called a sink and an unstable one is called a source. If $\det(A) = 0$ then 0 is called a degenerate equilibrium. One can view this classification in a plane $\alpha = tr(A)$, $\delta = \det(A)$. Such a diagram is called a bifurcation diagram.

2.2 Higher dimensions

Obviously this classification is too painful in higher dimensions. Instead, we focus on the stability or lack thereof. Even in the 2×2 case, if we started off with an eigenvector, we remained in that eigenspace. Akin to this phenomenon, we define as follows: Let A be an $n \times n$ real matrix. A subspace $E \subset \mathbb{R}^n$ is said to be invariant with respect to the flow e^{tA} if $etAE \subset E$ for all t.
In view of the Jordan canonical form, we define/recall generalised eigenvectors with generalised eigenvalue λ as vectors $v \neq 0$ such that $(A - \lambda I)^k v = 0$. The subspace of generalised eigenvectors corresponding to λ is called a generalised eigenspace. First we note that a generalised eigenspace is an invariant subspace. Indeed, bringing A to its Jordan canonical form, this follows easily. By the theorem of the Jordan canonical form, $\mathbb{R}^n$ is a direct sum of three kinds of subspaces: The ones with $Re(\lambda) < 0$: the stable subspace E^s, the unstable subspace E^u $Re(\lambda) > 0$, and the centre subspace E^c ($Re(\lambda) = 0$). These are all invariant subspaces. In fact, as $t \to \infty$, the flow takes E^s to 0 eventually and E^u to 0 as $t \to -\infty$. The analogous result for nonlinear systems is called the stable manifold theorem.