1 Recap

1. Classification of 2×2 equilibria.

2. Stable, unstable, and centre subspaces.

2 Nonlinear stability

We will largely study equations of the form $\vec{x}' = \vec{F}(\vec{x})$ where $\vec{x} \in \mathbb{R}^n$ (with the vector symbols usually omitted). This is an autonomous system of *dimension* n. An equilibrium point or a steady-state solution is a vector x_0 such that $F(x_0) = 0$. An equilibrium point can be isolated or non-isolated. If F is locally Lipschitz, by uniqueness, $x = x_0$ is the only solution with $x(t_0) = x_0$.
On paper, one can try to study things like $x' = x + t$ which are non-autonomous by reducing them to autonomous ones: Introduce a new variable $x_{n+1} = t$. Then $x_i' = F_i(x), x_{n+1}' = 1$. Unfortunately such a system *never* has equilibrium points even if the original one does (what is an example?). So this strategy does not always simplify matters.
Usually, assumptions are made on F so that the solution exists for all time for any or at least "most" initial data. Sometimes it is helpful to think of $\vec{x}(t)$ as a curve whose tangent vector is $\vec{F}(\vec{x})$. For this reason, $\vec{F}(\vec{x})$ is called a vector field and the solution is called an integral curve. Sometimes, $x_0 \to x(t)$ is called a time-t flow.
Def: The orbit $O(x_0)$ through x_0 is the solution of the ODE with $x(t_0) = x_0$. The positive orbit $O^+(x_0)$ is the solution with $t \geq t_0$. A periodic/closed orbit is a solution such that there exists a $T > 0$ such that for every t, $x(t + T) = x(t)$. The smallest such $T > 0$ is called the period. (What if the the infimum is 0?) Usually we exclude fixed points when we talk about periodic orbits. If a periodic orbit is isolated, then it is called a limit cycle.
Here are some properties about orbits.

Lemma 2.1. *Assume that $\vec{F}$ is locally Lipschitz.*

1. *Let $x(t)$ be an orbit passing through x_0. $x(t + c)$ is also a solution.*

2. *If $x(t_0) = y(t_1) = x_0$ and x, y are solutions, then $y(t) = x(t + t_0 - t_1)$.*

3. *Two orbits either coincide or do not intersect.*

4. *Suppose there exist $T > 0, t_0$ such that $x(t_0 + T) = x(t_0)$, then $x(t + T) = x(t)$ for all t.*

5. *There are no limit cycles for linear homogeneous autonomous systems $x' = Ax$. (For non-linear systems they can exist. Later on, we shall see that the Poincaré-Bendixon and Leinard's theorems give us sufficient conditions for $2D$ systems to have periodic solutions.)*

6. *Suppose x is a solution and $\lim_{t \to \infty} x(t) = \xi$ exists. Then ξ is an equilibrium point.*

Proof. 1. Easy (differentiate).

2. Again uniqueness and differentiation.

3. Uniqueness.

4. Uniqueness.

5. The explicit formula for x is $x(t) = e^{At}x_0$. Thus $x(t+T) = x(t)\,\forall\, t$ iff $(e^{AT}-1)x_0 = 0$. Either $x_0 = 0$ in which case it is an equilibrium point and does not count as periodic, or λx_0 also satisfies this condition for all λ and hence, for every periodic orbit, regardless of what neighbourhood we choose around it, there is another periodic orbit that intersects this neighbourhood.

6. Indeed, fix $h > 0$. Then $x(t + h) \to \xi$ as $t \to \infty$. By the mean-value theorem, $x_i(t + h) - x_i(t) = hf_i(x(\tilde{t}_i))$ where $\tilde{t}_i \in [t, t + h]$. As $t \to \infty$, so does $\tilde{t}_i$ and hence $f(\xi) = 0$.

□

Here are some examples.

1. $x' = x^2$: Only one equilibrium point. Unfortunately, the solution blows up in finite time.

2. $x' = \sin(x)$: Isolated equilibria. If we start at $x \in (2n\pi, (2n + 1)\pi)$, then $\sin(x) > 0$ and hence x is increasing but $|x'| \leq 1$. The solution exists for time (positive and negative). It can never be $(2n + 1)\pi$ (why?) Thus $\lim_{t\to\infty} x(t)$ exists. It must be $(2n + 1)\pi$. Likewise for odd multiples.