1 Recap

1. Definitions of limit cycles, equilibria.

2. Limit cycles do not exist for linear homogeneous autonomous systems. For autonomous systems in general, if the limit as $t \to \infty$ exists, the limit must be an equilibrium point.

2 Nonlinear stability

Here are some examples.

1. Consider $r' = \sin(r)$ where $r(0) > 0$, $\theta' = 1$. This system has no equilibrium points. However, let $x = r\cos(\theta)$, $y = r\sin(\theta)$. Then there are limit cycles in the $x - y$ plane. Indeed, if we start with $r = n\pi$, r will remain $n\pi$ whereas θ changes. There can be no other periodic orbit (because r is monotonic) nearby.

2. If one wants to model an elastic pendulum, one uses the Duffing equations: $x' = y, y' = \pm x - x^3 - \delta y$. For the negative sign, the only equilibrium is the origin. For the positive sign one also has $(\pm 1, 0)$.

3. $x' = -y + x(x^2 + y^2)$ and $y' = x + y(x^2 + y^2)$ has only the origin as the equilibrium point. (Indeed, $y = x(x^2 + y^2)$, $x = -y(x^2 + y^2)$ and hence x, y have the same sign and opposite sign!)

4. $x'' + xx' = 0$: Writing as a first order system, $x' = v, v' = -xv$, we see that all points on the line $v = 0$ are equilibrium points. So no isolated ones.

3 Liapunov stability

We confine ourselves mainly to Liapunov stability: An isolated equilibrium point $\bar{x}$ of $x' = f(x)$ is said to be Liapunov stable if given $\epsilon > 0$, there is a $\delta > 0$ such that for any solution with $\|x(t_0) - \bar{x}\| < \delta$, $\|x(t) - \bar{x}\| < \epsilon$ for all $t > t_0$. Otherwise it is said to be Liapunov unstable.
Asymptotic stability: An isolated equilibrium $\bar{x}$ is asymptotically stable if it is Liapunov stable and if $\|x(t_0) - \bar{x}\| < b$, then $\lim_{t\to\infty} \|x(t) - \bar{x}\| = 0$.
In the case of linear equations, $x' = Ax$, if $\det(A) \neq 0$, then 0 is the only equilibrium. If the eigenvalues have negative real parts, then it is asymptotically stable. If even one of them has a positive real part, it is unstable. If they have ≤ 0 real parts (with at least one zero real part), then it is stable (but not asymptotically so - why?).
One important idea is to approximate a nonlinear system by its Newton/first-order approximation. This is called linearisation. $f(\bar{x}+y) = f(\bar{x})+Df(\bar{x})y+O(\|y\|^2)$ (assume f is smooth for simplicity). We can now consider the linear system $y' = Ay$ and try to look at its stability properties. They may not always reflect that of the nonlinear system but sometimes they do. Here is an example of a theorem:

Theorem 1. *Suppose the eigenvalues of $A = Df(\bar{x})$ (where $\bar{x}$ is an equilibrium) all have negative real parts. Then the equilibrium point is asymptotically stable.*

This theorem is a special case of Perron's theorem:

Theorem 2. *Let A be a real $n \times n$ matrix whose eigenvalues all have negative real parts. Consider $x' = Ax + f(t, x)$ where f is continuous and satisfies $\|f(t, x)\| < \epsilon\|x\|$ for all $\|x\|\delta$ (δ is independent of t). Then any solution x with sufficiently small $x(0)$ exists for all $t \geq 0$ and $x(t)$ tends to 0 as $t \to \infty$.*

Note that even existence for all $t \geq 0$ is not straightforward! For instance, if $x' = x^2$, then $A = 0$ and the theorem above does not apply. (Indeed, in this case the solution blows up in finite time.) On the other hand, if $x' = -\mu x + x^2$ where $\mu > 0$, then the theorem does apply. But even more simply, $x' \geq -\mu x$ and hence x is bounded below (at least for finite positive time). Now if $0 < x(0) \leq \frac{\mu}{2}$, then $x' \leq -\frac{\mu}{2}x$ for small enough t. For all such t, $x \leq x_0 e^{-\mu/2t} \leq \frac{\mu}{2}$. Thus x stays in this interval for all $t \geq 0$ and indeed converges to 0 as $t \to \infty$. This is the main idea of the proof.