1 Recap

1. Examples of equilibria and limit cycles.
2. Liapunov stability and asymptotic stability.
3. Perron's theorem (statement and example):

   Theorem 1. *Let A be a real $n \times n$ matrix whose eigenvalues all have negative real parts. Consider $x' = Ax + f(t,x)$ where f is continuous and satisfies $\|f(t,x)\| < \epsilon\|x\|$ for all $\|x\|\delta$ (δ is independent of t). Then any solution x with sufficiently small $x(0)$ exists for all $t \geq 0$ and $x(t)$ tends to 0 as $t \to \infty$.*

2 Liapunov stability

Proof. Firstly note that $\|e^{tA}\| \leq Ke^{-\sigma t}$ for some $K > 1, \sigma > 0$ as we saw in the HW. Now by Cauchy-Peano, there exists a solution with any given initial data x_0 for a short time $[0, t_*]$. Moreover, we can prove (how) that $x(t) = e^{At}x_0 + \int_0^t e^{(t-s)A} f(s, x(s))ds$. By the hypothesis on f, there exists a $\delta > 0$ such that $\|f(t,x)\| \leq \frac{\sigma}{2K}\|x\|$ for all $\|x\| \leq \delta$. Now assume that $\|x_0\| \leq \frac{\delta}{K}$. We claim that for all such x_0, the solution exists for all time and satisfies $\|x\| \leq \delta$. Indeed, the solution exists on a maximal interval. Consider the set of all t for which $\|x\| \leq \delta$. This set obviously contains $[0, t_*]$ for some t_*. Consider the maximum such t_*. For all $t \leq t_*$, $\|x(t)\| \leq Ke^{-\sigma t}\|x_0\| + \frac{\sigma}{2}\int_0^t e^{-\sigma(t-s)}\|x(s)\|ds$. Multiplying by $e^{\sigma t}$ and using Gronwall, $e^{\sigma t}\|x\| \leq K\|x_0\|e^{\sigma/2t}$. Hence, $\|x\| \leq K\|x_0\|e^{-\sigma/2t} < \delta$ and hence the solution extends beyond t_* and satisfies $\|x\| \leq \delta$ beyond it as well. Thus t_* is not finite. □

Before we proceed, here is an important definition: An equilibrium $\bar{x}$ is called hyperbolic if all the eigenvalues of $Df(\bar{x})$ have non-zero real parts.
Here are some more examples:

1. $x' = -y + x(x^2 + y^2)$, $y' = x + y(x^2 + y^2)$. Here A has eigenvalues $\pm\sqrt{-1}$. So the theorem above does not apply. The linearisation at the origin has the origin as a stable (but not asymptotically stable) equilibirum point. However, considering $r = \sqrt{x^2 + y^2}$, $r' = r^3$ and hence if $r_0 > 0$, then r blows up in finite time. Thus the origin is unstable. On the other hand, if we consider $x' = -y - x(x^2 + y^2)$, $y' = x - y(x^2 + y^2)$, then $r' = -r^3$ and hence the origin is asymptotically stable.
2. The Duffing system with negative sign: $x' = y, y' = x - x^3 - \delta y$. The equilibria are $(0,0), (\pm 1, 0)$. The linearisation at $(0,0)$ has eigenvalues $\frac{-\delta \pm \sqrt{\delta^2+4}}{2}$. This point is linearly unstable when $\delta \geq 0$. At $(\pm 1, 0)$, the eigenvalues are $\frac{-\delta \pm \sqrt{\delta^2-8}}{2}$. Thus for $\delta > 0$, the eigenvalues have negative real parts and by the theorem above, these equilibria are asymptotically stable. If $\delta = 0$, the points are linearly stable but not linearly asymptotically stable. It is hard to analyse the nonlinear system.