1 Recap

1. Proof of Perron's theorem.
2. Couple of examples.

2 Liapunov stability

1. (Continued...) $x' = y, y' = x - x^3$. We claim that this system isnot stable at the origin. Indeed if it were, there exists $(0 < x_0 < 1, y_0 > 0)$ such that the solution exists on $[0, \infty)$ and satisfies $\|(x, y)\| < 1/2$. But in this regime, $y' \geq 0$ and hence $x' \geq y_0$ and hencce for large t we have a contradiction.

2. The van der Pol system: $x' = y, y' = \mu(x^2 - 1)y - x$. The origin is the only equilibrium. The linearisation has eigenvalues $\frac{-\mu \pm \sqrt{\mu^2 - 4}}{2}$. If $\mu > 0$ it is linearly (and hence nonlinearly) asymptotically stable. If $\mu < 0$ it is linearly unstable. When $\mu = 0$ we have a linear system which is stable. Again, the nonlinear analysis is tricky.

3. $x' = y, y' = -x + 2y - 2x^2$. Subtracting, $(x - y)' = (x - y) + 2x^2$ and hence $x - y = (x - y)_0 e^t + e^t \int_0^t 2x^2 e^{-s} ds > (x_0 - y_0)e^t$. Hence if $x_0 - y_0 > 0$, $x - y$ runs off to infinity. Thus this point is not stable.

4. Consider the Lorenz system (originally for atmospheric convection but later for simplified models of lasers, chemical reactions, etc - the original butterfly effect): $x' = \sigma(-x + y), y' = Rx - y - -xz, z' = -bz + xy$. Assume all constants are positive. If $R \leq 1$, the origin is the only equilibrium but for $R > 1$, $(\pm\sqrt{b(R-1)}, \pm\sqrt{b(R-1)}, R - 1)$ are two more points. For $R < 1$, the linearisation at the origin has negative eigenvalues (and hence asymptotically stable even nonlinearly) and when $R > 1$, one eigenvalue is positive. When $R > 1$, the eigenvalues at the other points are roots of $\lambda^3 + (\sigma + b + 1)\lambda^2 + (R + \sigma)b\lambda + 2\sigma b(R - 1) = 0$. It has a real eigenvalue for sure. When $R >> 1$, the derivative is positive and $f(0) > 0$. Thus there is only real eigenvalue and it is negative. It turns out that for large R, the real parts of the other eigenvalues is negative. Note that we transit from linear stability to instability at $R = 1$ at the origin. So $R = 1$ is called a bifurcation point.

5. Lastly, non-autonomous systems can be very strange: Consider $x' = A(t)x$ where $A(t) = \begin{bmatrix} -1 + \frac{3}{2}\cos^2 t & 1 - \frac{3}{2}\cos(t)\sin(t) \\ -1 - \frac{3}{2}\cos(t)\sin(t) & -1 + \frac{3}{2}\sin^2(t) \end{bmatrix}$. It turns out that (using power series or simply guessing) a basis of linearly independent solutions is $v_1 = e^{t/2}(-\cos(t), \sin(t))$ and $v_2 = e^{-t}(\sin(t), \cos(t))$. Thus the equilibrium point $(0, 0)$ is unstable and of saddle type. However, the eigenvalues of $A(t)$ are $\frac{-1 \pm \sqrt{7}i}{4}$ for all t!

3 Liapunov functions

To analyse the situation of non-hyperbolic equilibria (even for hyperbolic ones, it is not easy as we shall see), a powerful tool is the Liapunov function. Suppose 0 is an isolated equilibrium point and Ω is a neighbourhood that does not contain other equilibria (any other point can be analysed by translation).
Def: A C^1 function $V : \Omega \to \mathbb{R}$ satisfying $V(0) = 0$, $V(x) > 0$ for $x \in \Omega - \{0\}$, $\nabla V.F \leq 0$ in Ω is called a Liapunov function.
Here is an important theorem: Suppose a Liapunov function V exists. Then the origin is stable. If $\nabla V.F < 0$ on $\Omega - \{0\}$, then 0 is asymptotically stable.