1 Recap

1. Examples of stability and instability.

2. Liapunov function definition.

2 Liapunov functions

To analyse the situation of non-hyperbolic equilibria (even for hyperbolic ones, it is not easy as we shall see), a powerful tool is the Liapunov function. Suppose 0 is an isolated equilibrium point and Ω is a neighbourhood that does not contain other equilibria (any other point can be analysed by translation).
Def: A C^1 function $V : \Omega \to \mathbb{R}$ satisfying $V(0) = 0$, $V(x) > 0$ for $x \in \Omega - \{0\}$, $\nabla V.F \leq 0$ in Ω is called a Liapunov function.
Here is an important theorem: Suppose a Liapunov function V exists. Then the origin is stable. If $\nabla V.F < 0$ on $\Omega - \{0\}$, then 0 is asymptotically stable.

Proof. The point is that $\frac{dV}{dt} = \nabla V.x = \nabla V.F \leq 0$ and hence V is decreasing along the trajectory. Suppose we consider a sphere C of radius ϵ (where ϵ is given but small) such that it and its interior lies entirely within Ω, and let m be the minimum of V on this sphere. For all $\|x\| \leq \delta$, $V \leq \frac{m}{2}$. If $\|x(t_0)\| \leq \delta$, then originally $V \leq \frac{m}{2}$. If at some point of time, $\|x(t)\|$ crosses C, then V would have to be $> \frac{m}{2}$ which is a contradiction. Hence $\|x(t)\|$ is bounded (and hence the maximal interval is all of $\mathbb{R}$) and bounded by the given ϵ and hence 0 is stable.
If strict inequality holds, then V is strictly decreasing. Now suppose $L = \lim_{t\to\infty} V(x(t))$ is positive. Then consider a smaller sphere $\tilde{C}$ such $V(x) \leq \frac{L}{2}$ inside the corresponding ball. By assumption, $V(x(t)) \geq L$ along the trajectory and hence cannot intersect $\tilde{C}$. Thus $\nabla V.f \geq -k$ along the trajectory which provides a contradiction (why?)
Now we can show that $\lim_{t\to\infty} \|x(t)\| = 0$. Indeed, if there is a sequence of times $t_n \to \infty$ for which $\lim_{n\to\infty} x(t_n) = x \neq 0$, then $\lim_{n\to\infty} V(x(t_n)) = V(x) > 0$ which is a contradiction. □

Here is an instability result (Chetaev's theorem)

Theorem 1. *Suppose there is a C^1 function $V : \Omega \to \mathbb{R}$ satisfying $V(0) = 0$, and there exists a $\tilde{\epsilon} > 0$ so that in every neighbourhood of size $< \tilde{\epsilon}$ of 0, there is a non-empty set where $V > 0$ and $\nabla V.\vec{F} > 0$ on the region $V > 0$, then 0 is unstable.*

Proof. What we want to show is that there exists a sequence $(x_0)_n \to 0$ and a sequence t_n such that $\|x(t_n)\| \geq \tilde{\epsilon}$ or for infinitely many terms of the sequence, the solution does not exist for all positive time: Choose a closed ball $B_n \subset \Omega$ of size $\epsilon_n = \frac{1}{n} < \tilde{\epsilon}$ centred at 0. There exists an $(x_0)_n \in Int(B)$ such that $V((x_0)_n) > 0$. Consider a solution with $x_n(0) = (x_0)_n$. We see that $t \to V(x_n(t))$ is increasing for $t \geq 0$. Thus the positive orbit is confined to the set $V > 0$. The claim is that this orbit crosses ∂B at some finite time (note that if the orbit does not exist for all $t \geq 0$, it is not stable by definition anyway). Indeed, suppose note. Then it remains in a compact set that does not contain the origin.

Thus $\nabla V.\vec{F} \geq m > 0$ on the orbit and that is a contradiction (because it implies V can grow in an unbounded manner). $\square$