1 Recap

1. Suppose a Liapunov function V exists. Then the origin is stable. If $\nabla V.F < 0$ on $\Omega - \{0\}$, then 0 is asymptotically stable.

2. (Chetaev's theorem)

   Theorem 1. *Suppose there is a C^1 function $V : \Omega \to \mathbb{R}$ satisfying $V(0) = 0$, and there exists a $\tilde{\epsilon} > 0$ so that in every neighbourhood of size $< \tilde{\epsilon}$ of 0, there is a non-empty set where $V > 0$ and $\nabla V.\vec{F} > 0$ on the region $V > 0$, then 0 is unstable.*

2 Liapunov functions

We now consider a few examples.

1. Consider $x' = y, y' = x - x^3$. Note that $V = xy$ satisfies $\nabla V.\vec{F} = (y, x).(y, x - x^3) = y^2 + x^2 - x^4 > 0$ when $x^2 < 1$ (and $(x, y) \neq 0$). Thus by the above theorem, the origin is unstable.

2. Consider $x' = -y^3, y' = x^3$. The only equilibrium is the origin and there $Df(0) = 0$. Thus Perron is not applicable. Consider $V = x^4 + y^4$. Then $V(0) = 0$, $V > 0$ away from the origin, and $\nabla V.\vec{F} = (4x^3, 4y^3).(-y^3 + x^3) = 0$. Thus the origin is stable. Actually in this case, the orbits stay on $V = c$. Thus the origin is not asymptotically stable.

3. Consider $x' = -2y + yz, y' = x - xz, z' = xy$. The origin is an equilibrium. The eigenvalues of the Jacobian at the origin are $0, \pm 2i$ and hence Perron is not applicable. Consider $V = c_1x^2 + c_2y^2 + c_3z^2$. Then if $c_1 = c_3 > 0$ and $c_2 = 2c_1$, $V(x) > 0$ for $x \neq 0$ and $\nabla V.\vec{F} = 0$. Thus the orbits lie on ellipsoids (and are stable but not asymptotically stable). The Liapunov theorem is not applicable here! The reason is that the equilibria are: $(0, 0, z)$, $(0, y, 2)$, and $(x, 0, 1)$, which are not isolated!

4. Consider $x' = -2y + 2yz - x^3, y' = x - xz - y^3, z' = x^2z - z^3$. The origin is an isolated equilibrium (because the other equilibria arise as zeroes of one-variable polynomials). The Jacobian matrix again has the same eigenvalues as the previous example. This time $V = x^2 + 2y^2 + z^2$ works to prove asymptotic stability because $\nabla V.\vec{F} < 0$.

5. $x' = x^2 + 2y^5, y' = xy^2$ has the origin as the only equilibrium. The linearisation is 0. Now consider $V = x^2 - y^4$ (which is 0 at the origin). Now $\nabla V.F = (2x, -4y^3).(x^2 + 2y^5, xy^2) = 2x^3 + 4xy^5 - 4xy^5 = 2x^3 > 0$ when $x > 0$. Now $V > 0$ when $x^2 > y^4 \geq 0$. Hence by Chetaev, the origin is unstable.