1 Recap

1. Examples of stability and instability. Mistake in Nandakumaran's book.

2 Invariant sets and manifolds

A set $S \subset \mathbb{R}^n$ is said to be invariant under $x' = f(x)$ if for any $x_0 \in S$, $x(t) \in S \; \forall \, t \in \mathbb{R}$ and positively invariant if $x(t) \in S \; \forall \, t \in [0, \infty)$.
Sometimes these sets can possess more structure. Here are some examples.

1. $x' = x, y' = -y + x^2$. The only equilibrium is the origin. The linearisation has eigenvalues ± 1. So the stable subspace and unstable subspace each have dimension 1. We can actually explicitly solve this system to get $x = x_0 e^t, y = y_0 e^{-t} + \frac{x_0^2 e^{-t}(e^{3t}-1)}{3}$. Eliminating t, $x(y - x^2/3) = x_0(y_0 - x_0^2/3)$. These are invariant subsets of the ODE. Moreover, if $x_0 = 0$, $x = 0$ and $y \to 0$ as $t \to \infty$. This set is called a "stable manifold" (what is a manifold? Whatever it is, it is a generalisation of a "smooth regular" surface in $\mathbb{R}^3$. It is supposed to be a nonlinear generalisation of a vector space.) If $y_0 = \frac{x_0^2}{3}$, then $y = \frac{x^2}{3}$ is an invariant subset. Moreover, every point on this subset goes to ∞ as $t \to \infty$ (actually, more precisely it goes to 0 as $t \to -\infty$).

2. We can take a similar example in three dimensions: $x' = -x, y' = -y + x^2, z' = z + x^2$. The linearisation at the origin (the only equilibrium) has eigenvalues $-1, -1, 1$ (so hyperbolic). Thus the stable subspace is 2-dimensional and the unstable one 1-dimensional. Upon solving, $x = x_0 e^{-t}, y = y_0 e^{-t} + x_0^2(e^{-t} - e^{-2t}), z = z_0 e^t + \frac{x_0^2 e^t (1 - e^{-3t})}{3}$. Thus $\|(x, y, z)\| \to 0$ (as $t \to -\infty$) iff $x_0 = y_0 = 0$. On the other hand $\|(x, y, z)\| \to 0$ iff $z_0 + \frac{x_0^2}{3} = 0$. These sets are invariant (why?)

3. Consider a previous example: $x' = -2y + yz, y' = x - xz, z' = xy$. Then $x^2 + 2y^2 + z^2 = a^2$, $x^2 + y^2 + z = b$ are invariant sets. The origin is not hyperbolic.

We want to say roughly that under some conditions (hyperbolicity), there is a nonlinear version of the stable-unstable subspaces theorem. To state this theorem rigorously, we need to know what a manifold is. The simplest nonlinear generalisation of a subspace of $\mathbb{R}^n$ is the graph of a function. Indeed, this is what manifolds are modelled after.
Def: An n-dimensional C^r (embedded sub)manifold in $\mathbb{R}^N$ is a set $S \subset \mathbb{R}^N$ (with the induced topology) such that given any point $p \in S$, there exists a neighbourhood $p \in U \subset \mathbb{R}^N$ such that $S \cap U$ is the homeomorphic image of a C^r map $f : V \subset \mathbb{R}^n \to \mathbb{R}^N$ (where V is open) such that Df has rank-n everywhere. Here r can be ∞ in which case it is called a smooth manifold. r can also be 0 (and we drop the derivative condition) in which case it is called a topological (embedded sub)manifold.