1 Recap

1. Invariant sets and examples.
2. Definition of (embedded sub)manifolds in $\mathbb{R}^N$.

2 Invariant sets and manifolds

The f's are called C^r local parametrisations (and their inverses are called 'charts'). It turns out using the inverse function theorem that in fact one can write a C^r (when $r \geq 1$) manifold locally as C^r graphs over some coordinate-axes in $\mathbb{R}^N$. Indeed, just for some practice let's prove this fact. Firstly, let's recall the inverse function theorem:
Suppose $F : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ is a C^r map (where Ω is open and $\infty \geq r \geq 1$) and DF_p is invertible, then there exist neighbourhoods $p \in U$ and $f(p) \in V$ such that f takes U to V homeomorphically and the inverse is C^r (it is a local C^r diffeomorphism).
Given this theorem, n rows in $Df_{f^{-1}p}$ are linearly independent. Suppose they are $i_1, \ldots, i_n$. Consider the map $T(u_1, \ldots, u_n) = (f_{i_1}, \ldots, f_{i_n})$. This map is C^r and $DT_{f^{-1}p}$ is invertible. Hence by the inverse function theorem, locally, u's are C^r functions of x_{i_k}'s. Thus the rest of the f's are C^r functions of x_{i_k}'s locally.
Here are examples and non-examples of manifolds in $\mathbb{R}^N$:

1. The graph of $y = |x|$ is not a C^1-manifold in $\mathbb{R}^2$: Indeed, if it were, then near the origin, y is a C^1 function of x (why?) and that is a contradiction.
2. The shape of the letter X is not even a topological manifold in $\mathbb{R}^2$ (why?)
3. The graph of any smooth function $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth n-dimensional manifold in $\mathbb{R}^{n+1}$.
4. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is smooth, $f^{-1}(0)$ is non-empty, and $\nabla f \neq 0$ at every point of $f^{-1}(0)$, then $f^{-1}(0)$ is an $n-1$-dimensional manifold in $\mathbb{R}^n$ (why? This uses the implicit function theorem). For instance, the sphere is a manifold.
5. An open subset of $\mathbb{R}^n$ is an n-dimensional manifold.

We now define the tangent space $T_pM \subset \mathbb{R}^N$ of a C^r ($r \geq 1$) manifold in $\mathbb{R}^N$: It is the subspace spanned by $\frac{\partial f}{\partial u_i}(f^{-1}(p))$. This subspace does not depend on the choice of f (why?)
Now we can state the stable manifold and the Hartman-Grobman theorems:

Theorem 1. *Consider $x' = f(x)$ where $f : \Omega \to \mathbb{R}^n$ is C^1 and Ω is a neighbourhood of the origin which is an equilibrium point. Let $Df(0)$ have k eigenvalues with strictly negative real parts and $n-k$ with strictly positive real parts (so 0 is hyperbolic). Then the following hold.*

1. *There exists a C^1-k-dimensional manifold S that is tangent to the stable subspace at 0 such that S is positively invariant and every point is stable (that is $\lim_{t\to-\infty} x = 0$).*
2. *There exists a C^1-$n-k$-dimensional manifold U that is tangent to the unstable subspace at 0 such that S is negatively invariant and every point is unstable (that is $\lim_{t\to-\infty} x = 0$).*