1 Recap

1. Manifolds and tangent spaces.
2. Stable manifold theorem.

2 Invariant sets and manifolds

Theorem 1. *Consider $x' = f(x)$ where $f : \Omega \to \mathbb{R}^n$ is C^1 and Ω is a neighbourhood of the origin which is a hyperbolic equilibrium point. Let $A = Df(0)$. There exists a homeomorphism of a neighbourhood $0 \in U$ to $0 \in V$ such that for each $x_0 \in U$, there is an open interval $I_0 \subset \mathbb{R}$ containing 0 such that $H(x(t)) = e^{tA}H(x_0)$ for all $t \in I_0$. That is, H maps the orbits near the origin to the orbits of the linearisation near the origin and preserves the parametrisation in time.*

3 Phase plane analysis aka the energy method

Many equations in physics are of the form $\vec{x}'' + \nabla V = 0$. These correspond to conservative forces. Taking inner product with $\vec{x}'$ on both sides and integrating, $\frac{\|x\|^2}{2} + V = E$. This is the law of conservation of energy. While we may not always be able to solve for $\vec{x}$ "explicitly" from this equation, we can at least gain some information. (If we have "enough" conserved quantities (which happens when we have enough symmetry - Noether's theorem), we can in fact solve "explicitly". This is the content of integrable systems.)

Here is an application of the fact that we have a conservation law. Assuming that $V \geq -C$ (which is a reasonable assumption that is satisfied for instance in spring block systems), $\|x\| \leq C$. Thus there is no finite-time blowup and hence solutions exist for all (positive and negative) time.

Here are some examples.

1. $x'' + kx = 0$ with $k > 0$. Note that $(x')^2 + kx^2 = 2E$. Thus, if we take $v = x', x$ as variables (converting to a first order system), we see that the orbits lies on ellipses. This does not immediately imply periodicity but even without solving explicitly, we can deduce periodicity with some analysis.

2. $x'' + k\sin(x) = 0$ with $k > 0$ (the actual pendulum). Again, $(x')^2 + 2k(1 - \cos(x)) = 2E$. Now $V(x) = 2k(1 - \cos(x)) \geq 0$ and hence $E \geq 0$ and the solution exists for all time. Note that $V(x) = 0$ iff $x = 2n\pi$. and $V(x) = 2k$ iff $x = (2n-1)\pi$. We expect that for small energy, the pendulum will oscillate. We have a few cases.

   (a) $E = 0$. Here, $V = 0$ and we are at equilibrium.

   (b) $0 < E < 2k$: Here, x lies in an interval around the equilibrium $2n\pi$. The maximum it can deviate from equilibrium is b where $V(2n\pi + b) = E$. These orbits are periodic:
   Proof: Define b by $V(2n\pi + b) = 0$ where $(2n-2)\pi < x_0 < (2n+2)\pi$. Note that $V \leq E$ with equality holding iff $V = 0$. Since $0 < E < 2k$,

V is never equal to $2k$ along the solution. Thus the solution lies within $[2n\pi - b, 2n\pi + b] \subset ((2n-1)\pi, (2n+1)\pi)$. Now suppose $2n\pi + b > x_0 > 2n\pi$ and $x'(0) > 0$ (there are a few other possibilities for initial data but the idea of the proof is the same and so we will leave the rest as an exercise).

Now $x'(0) > 0$ and hence initially and for a short period of time, $x' = \sqrt{2E - V}$ and x is increasing. We claim that there exists a first time $t_0 > 0$ such that $x'(t_0) = 0, x(t_0) = 2n\pi + b$. Suppose not. Then $x' > 0$ for all t. Thus as $t \to \infty$ either $\lim x'(t)$ is strictly positive which is a contradiction (because x will then go off to infinity, but as we argued $x \in [2n\pi - b, 2n\pi + b]$), or the limit is 0. In that case, the limit of x (which exists because supposedly x is increasing) is $2n\pi + b$. Thus $\lim x'' = -k\sin(2n\pi + b)$. Now this limit must be 0 (if not, x' runs off to ∞ or $-\infty$). Thus $2n\pi + b$ is a multiple of π which is a contradiction to the assumption that $0 < E < 2k$. Thus there is a first time t_0 when $x(t_0) = 2n\pi + b, x'(t_0) = 0$.

Now we claim that for $t_0 + \epsilon > t > t_0$, $x'(t) < 0$. If not, $x'(t_0 + h_n) = 0$ for all n and $h_n \to 0$. This means $x''(t_0) = 0$ and again that means $\sin(2n\pi + b) = 0$ which is a contradiction.

Thus $x' = -\sqrt{2E - 2V}$ for some time after t_0. Again, as above, we can argue that there exists a first time t_1 when $x(t_1) = 2n\pi - b$ and $x'(t_1) = 0$. Then we continue similarly to show that there is a first time $t_2 > t_1$ such that $x(t_2) = 2n\pi + b$, $x'(t_2) = 0$. This means that at some time $T > t_1$ we should have reached $(x(0), x'(0))$ which proves that the orbit is periodic.

(c) $E = 2k$: To be continued....