1 Recap

1. Hartman Grobman theorem
2. Energy method for phase plane analysis (harmonic oscillator and the pendulum with $E = 0, 0 < E < 2k$ - here the equilibrium is stable but not asymptotically so (why?)).

2 Phase plane analysis aka the energy method

Here are some examples.

1. $x'' + k\sin(x) = 0$ with $k > 0$ (the actual pendulum). Again, $(x')^2 + 2k(1-\cos(x)) = 2E$. Now $V(x) = 2k(1-\cos(x)) \geq 0$ and hence $E \geq 0$ and the solution exists for all time. Note that $V(x) = 0$ iff $x = 2n\pi$. and $V(x) = 2k$ iff $x = (2n-1)\pi$. We expect that for small energy, the pendulum will oscillate. We have a few cases.
   (a) $E = 2k$: The orbits are not periodic but are bounded and the equilibrium points are unstable. (HW)
2. $x'' - x + x^3 = 0$: Here $E = \frac{1}{2}(x')^2 - \frac{x^2}{2} + \frac{x^4}{4}$. Note that $V \geq \frac{-1}{4}$ (and hence the solution exists for all time) is symmetric and $E \geq -\frac{1}{4}$. Again we have a few cases.
   (a) $E = -\frac{1}{4}$: Equilibrium $(\pm 1, 0)$.
   (b) $-\frac{1}{4} < E < 0$: Periodic solutions surrounding each of the two equilibrium points of length $2b$ where $V(\pm 1 \pm b) = E$. (HW)
   (c) $E = 0$: Two orbits (with $x(0) > 0$ and $x(0) < 0$) approaching the equilibrium $(0,0)$ as $t \to \pm\infty$. In fact, if $x(0) = x_0 > 0$ and $x'(0) \geq 0$ (for instance), then $x = 2\sqrt{2}{y_0}^{1/2}\frac{e^t}{1+y_0e^t}$ where $y_0 = \frac{\sqrt{2}-\sqrt{2-x_0^2}}{\sqrt{2}+\sqrt{2-x_0^2}}$. These are called homoclinic orbits. (HW)
   (d) $E > 0$: We still get periodic orbits but restricted to $[-b, b]$ where $V(\pm b) = E$ and $b > \sqrt{2}$. (HW)

3 Periodic orbits

We study $\vec{x}' = \vec{F}(\vec{x})$ in two dimensions only for periodic orbits. That is, $x' = f(x,y), y' = g(x,y)$ where f, g are C^k ($k \geq 2$) functions. We first need the notion of Poincaré index. We are interested in smooth parametrised paths in $\mathbb{R}^2$, i.e., smooth maps $\gamma : [a,b] \to \mathbb{R}^2$. A piecewise smooth path is continuous and smooth away from finitely many points. A smooth regular path is a smooth path such that $\gamma' \neq 0 \ \forall \ t$. (Likewise piecewise regular.) A closed path is one for which $\gamma(a) = \gamma(b)$. A smooth simple closed path or a smooth Jordan curve is a closed path that is $1-1$ everywhere except the endpoints. The image of a path is sometimes called a curve (but terminology is often abused and paths are also called curves).

The famous Jordan curve theorem implies that piecewise smooth Jordan curves divide

the plane into two connected components (one of which is unbounded) such that the curve is the boundary of both. So we can talk about the "domain" that is "inside" the Jordan curve.
The aim is to know when periodic orbits exist and when they don't. Note that along a periodic orbit, the tangent vector "rotates" by at least 2π. This means it is perhaps important to measure the (anticlockwise) "rotation index" of a vector field along a curve. Note that the vector field of interest to us will be $\vec{v} = f\hat{i} + g\hat{j}$. Let us look at some examples.

1. Let $\gamma : [0, 2\pi] \to \mathbb{R}^2$ be $\gamma(t) = (\cos(t), \sin(t))$. Consider $\vec{v} = (x, y)$. We can explicitly solve $x' = x, y' = y$ to get $(x, y) = (x_0, y_0)e^t$. Clearly there are no periodic orbits. Indeed, the picture of the vector field looks like it is "diverging away" from the origin. Now the rotation of this vector field along γ is 2π of course. (We haven't defined it rigorously but we can "see" it.) Note that if we consider $-\vec{v}$, the rotation would be still be 2π (why?).
   Consider $\vec{v} = (y, -x)$. Now $x' = y, y' = -x$ and hence $(x, y) = (x_0, y_0)e^{-it}$. Thus every orbit away from $(0, 0)$ is periodic. Indeed, the vector field "curls" around the origin. It has rotation 2π. (By the way, this "curl" business seems to suggest that Green's theorem might play a role and indeed it does.)

2. Consider a unit circle around $(-2, 0)$ and $\vec{v} = (x, y)$. What is the rotation? (It is zero!)