1 Recap

1. Energy method (Duffing's equation).
2. Definition of paths, curves, Jordan curves, etc. Examples for illustrating rotation index (not defined yet).

2 Periodic orbits

It seems that a periodic orbit always encloses an equilibrium. Indeed, we shall see that this is true. We now define the Poincaré index of a smooth vector field with isolated equilibrium points with respect to a piecewise smooth piecewise regular Jordan curve: Suppose γ does not pass through any equilibrium point. The index is $I_{\vec{v}}(\gamma) = \frac{1}{2\pi}\int_a^b \frac{v_1 v_2' - v_2 v_1'}{v_1^2 + v_2^2} ds$ where γ is parametrised in the "anticlockwise direction", i.e., $\hat{k} \times \gamma'$ points inside.
This definition is reasonable because we can prove the existence of a continuous piecewise smooth function $\theta : [a, b] \to \mathbb{R}$ such that $\vec{v}(\gamma(t)) = \|\vec{v}\|(\gamma(t))(\cos(\theta), \sin(\theta))$. Indeed, the assumption of no equilibria along the curve implies this fact by using the notion of a continuous lift (just as in the Prüfer substitution). Now $\theta' = \frac{v_1 v_2' - v_2 v_1'}{v_1^2 + v_2^2}$ and hence the index is precisely the "number of times" the vector field "winds" around. We can now calculate the index for the above examples (as well as say for (x^2, y^2) around the unit circle) and verify that our expectations are correct.
Theorem: Let γ be a piecewise regular piecewise smooth Jordan curve such that it and its interior do not contain any equilibria. Then $I_{\vec{v}}(\gamma) = 0$.
Proof: By Green's theorem, we see the result holds. □
In fact, the above result holds for any bounded domain whose boundary is a finite union of piecewise regular piecewise smooth Jordan curves provided they are parametrised in the "right" direction. Using this observation, we can talk of the index of an isolated equilibrium without reference to any specific Jordan curve (because we can prove that the answer does not depend on the curve).
We can prove a more general result: If γ is a piecewise smooth piecewise regular Jordan path whose interior contains only finitely many equilibria of $\vec{v}$ (that does not vanish on the image of γ), then the index of $\vec{v}$ is the sum of indices of each equilibrium.
Now note that the index can be defined even if the vector field is defined on the image of the Jordan curve. Using this observation, we can ask what the index of a curve with respect to its own tangent vector is, i.e., how much does the tangent vector rotate?

Theorem 1. *Let γ be a C^1 Jordan curve which is regular. Then $I_{\gamma'}(\gamma) = 1$.*