1 Recap

1. Poincaré index.
2. $I_{\vec{v}}(\gamma) = 0$ if $\vec{v}$ has no equilibria inside.
3. Definition of $I_{\gamma'}(\gamma)$ and statement that it is 1.

2 Periodic orbits

Theorem 1. *Let γ be a C^1 Jordan curve which is regular and $\gamma'(a) = \gamma'(b)$. Then $I_{\gamma'}(\gamma) = 1$.*

Proof. Firstly, we can assume that $\gamma : [0, l] \to \mathbb{R}^2\ 0$ has unit speed (why?) Secondly, by translating and reparametrisation (does not change the index - why?), we can assume that γ is in the upper half-plane, and $\gamma(0)_2 = \gamma_2(l) = 0$ (with $\gamma_2 > 0$ otherwise). The idea is to compare the rotation of the tangent vector with that of the secant/chord around the x-axis. So consider the secant vector field $X(s,t)$ defined on the triangular region $0 \leq s \leq t \leq l$ as $X(s,s) = \gamma'(s)$, $X(s,t) = \frac{\gamma(t)-\gamma(s)}{\|\gamma(t)-\gamma(s)\|}$ when $t > s,\ (s,t) \neq (0,l)$ and $X(0,l) = -\gamma'(0)$. This vector field is continuous (why?). Let $\theta(s,t)$ be the angle made by $X(s,t)$ with the positive x-axis. (It is well-defined and continuous by the lifting property.) Choose the initial point such that $\theta(0,0) = 0$. Clearly, $\theta(0,t)$ varies from 0 to π as t varies from 0 to l. On the other hand $\theta(s,l)$ varies from π to 2π. Thus $\theta(s,s)$ changes by 2π on $[0,l]$. This means the index is 1. □