1 Recap

1. Definition of $I_{\gamma'}(\gamma)$ and proved it is 1 for regular Jordan curves with $\gamma'(a) = \gamma'(b)$. (Note that we made a mistake in defining the secant vector field. It should have been $\frac{\gamma(t)-\gamma(s)}{\|\gamma(t)-\gamma(s)\|}$.

2 Periodic orbits

As a consequence, we see that if C is a periodic orbit, then the index is 1 and if the equilibria are isolated, then C contains only finitely many equilibria whose indices sum up to 1 (and hence in particular, there is at least one equilibrium inside C). But this criterion is only specific to two dimensions.
Example: Consider $x' = y, y' = -x, z' = 1 - x^2 - y^2$. This has no equilibria and yet $(\cos(t), \sin(t), c)$ are periodic orbits
Now we want criteria for the existence or the lack thereof of periodic orbits. (Obviously $v = (f, g)$ must have equilibria. This is already an easy condition to verify.)
Bendixon's criterion: If $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$ has a definite sign in Ω (assume (f, g) is C^1), then Ω has no periodic orbit.
Proof: If Ω has a periodic orbit C, note that on $[0, T]$, C is a Jordan curve that is regular. Let D be interior to C. Apply Green to see that $\int_C (f dy - g dx)$ has a sign but also vanishes - a contradiction.
Here is the Poincaré-Bendixon theorem:

Theorem 1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain together with its boundary. Suppose it does not contain any equilibrium (typically we think of Ω as an " annular" region). If C is an orbit in Ω for $t \geq t_0$, then either C is periodic or it "tends" to a periodic one as $t \to \infty$ (that is, the distance between $\gamma(t)$ and the periodic one goes to 0).*

In other words, either orbits are unbounded or bounded. If you take a bounded orbit, either it converges to equilibrium or to a periodic orbit. Here is an example:
Let $x' = -y + x(1 - x^2 - y^2)$, $y' = x + y(1 - x^2 - y^2)$. We want to consider periodic orbits. To this end, firstly, the only equilibrium is $(0, 0)$. Consider an annulus $\Omega = \{\frac{1}{2} \leq r \leq \frac{3/2}{\}}$. It does not contain equilibria but "surrounds" one. If we find some orbit positively contained in this annulus, then there is a periodic orbit. To this end, let $r^2 = x^2 + y^2$. Then $r' = r(1 - r^2)$. We can easily see that $r = 1$ is a periodic orbit. We can also deduce the existence of a periodic orbit using Poincare'e-Bendixon. (By the way, we can easily see using Bendixon's criterion that there are no periodic orbits very close or very far from the origin.) In fact, we can solve explicitly to conclude that $r = (1 + c\exp(-2t))^{-1/2}$ where $c = \frac{1-r_0^2}{r_0^2}$. Thus every orbit spirals towards the only periodic one.
For the particular case of second order equations, we have a better result due to Leinard: Consider $x'' + f(x)x' + g(x) = 0$ where f, g satisfy

1. f, g are C^1 functions on $\mathbb{R}$,

2. g is odd, $g > 0$ for $x > 0$, and f is even.

3. The odd function $F(x) = \int_0^x f(s)ds$ has exactly one positive zero a, $F < 0$ on $(0, a)$ and > 0 on (a, ∞), F is increasing in (a, ∞), and $F(x) \to \infty$ as $x \to \infty$.

Then there is a unique periodic orbit surrounding the only equilibrium point $(0, 0)$. Moreover, every other orbit approaches this one as $t \to \infty$.
Example: $x'' + \mu(x^2 - 1)x' + x = 0$ with $\mu > 0$ produces a limit cycle.

3 Stable manifold theorem - Proof

Here is a high-level idea: Basically, $x' \approx Ax$ where $A = Df_0$. Now collecting the generalised eigenspaces with negative and positive real parts (respectively), we see that $B = CAC^{-1}$ is block diagonal (with *Real* blocks P, Q by the *Real* Jordan canonical form theorem) and $y = Cx$ satisfies $y' \approx By$. Now roughly speaking, we want to solve for $y_{k+1}, \ldots, y_n$ in terms of $y_1, \ldots, y_k$ (such that the solution is close to the plane $y_{k+1} = 0 = \ldots$). The idea is to go through the usual proof of existence by successive approximations carefully (using the Duhamel formula treating the error term as the forcing term), and start with initial data such that we expect the solution to go to 0 as $t \to \infty$. Then it turns out that such an initial data set is naturally a graph locally near the origin over $y_1, \ldots, y_k$. We then prove that if we do not start on this set, we cannot go to infinity. Using this we prove that S is invariant. For U, we simply change the direction of time and apply the same argument.

To go a little further into detail, note that $y' = By + G(y)$ where because F is C^1, for any $\epsilon > 0$, there exists a $\delta > 0$ such that if $\|x\| \leq \delta$, $\|y\| \leq \delta$, $\|G(y) - G(x)\| \leq \epsilon \|x - y\|$. We shall choose ϵ later.