1 Recap

1. Bendixon, Poincaré-Bendixon, Leinard's theorems.
2. Started the proof of the stable manifold theorem.

2 Stable manifold theorem - Proof

To go a little further into detail, note that $y' = By + G(y)$ where because F is C^1, for any $\epsilon > 0$, there exists a $\delta > 0$ such that if $\|x\| \leq \delta$, $\|y\| \leq \delta$, $\|G(y) - G(x)\| \leq \epsilon\|x - y\|$. We shall choose ϵ later. Let $U(t)$ be block diagonal with the upper block being e^{Pt} and everything else being zero. Likewise $V(t)$ has e^{Qt} on the lower block. Note that $e^{Bt} = U + V$. The Duhamel principle states that

$$y(t) = U(t)y(0) + \int_0^t U(t-s)G(y(s))ds + V(t)y(0) + \int_0^t V(t-s)G(y(s))$$

which (for ease of grouping the stable and unstable terms) is

$$y(t) = U(t)y(0) + \int_0^t U(t-s)G(y(s))ds + V(t)(y(0) + \int_0^\infty V(-s)G(y(s))ds) - \int_t^\infty V(t-s)G(y(s))ds.$$

Note that the integral from 0 to ∞ is expected to exist (why?) All the terms except for $V(t)(y(0) + \int_0^\infty V(-s)G(y(s))ds)$ are expected to go to 0 as $t \to \infty$. Thus for stability to hold, we better require $(y(0) + \int_0^\infty V(-s)G(y(s))ds)$ to be constant and have the last few components as 0. Indeed, we turn this around and state our mission as solving the integral equation

$$y(t,a) = U(t)a + \int_0^t U(t-s)G(y(s,a))ds - \int_t^\infty V(t-s)G(y(s,a))ds$$

with $a_{k+1} = a_{k+2} = \ldots = 0$. (We want to choose
$y_i(0) = \psi_i(a_1, \ldots, a_k) = -\int_0^\infty V(-s)G(y(s, a_1, \ldots, a_k, 0, \ldots))ds$ when $i \geq k+1$.) In other words, the initial values of y_i (for $i \geq k+1$) are hopefully graphs of functions of $y_1(0), \ldots$. Thus if we manage to solve the equation on $\|(y_1, \ldots, y_k)(0)\| < \delta'$ for some δ', then on this set, we have a manifold of dimension k given as a graph. We shall prove that this manifold is invariant and that it is indeed the stable manifold.
To solve the integral equation for small values of $(y_1(0), \ldots, y_k(0))$, we use an iteration with $y^{(0)} = 0$. Then define

$$y^{(j+1)}(t,a) = U(t)a + \int_0^t U(t-s)G(y^{(j)}(s,a))ds - \int_t^\infty V(t-s)G(y^{(j)}(s,a))ds.$$

As in the usual proof of existence (Picard's theorem), we want to prove that successive iterates get close to each other. Note that

$$\begin{aligned}\|y^{(j+1)}(t,a) - y^{(j)}(t,a)\| &\leq \int_0^t \|U(t-s)\|\|G(y^{(j)}(s,a)) - G(y^{(j-1)}(s,a))\|ds \\ &+ \int_t^\infty \|V(t-s)\|\|G(y^{(j)}(s,a)) - G(y^{(j-1)}(s,a))\|ds. \qquad (1)\end{aligned}$$

At this point note that $\|U(t)\| \leq Ke^{-(\alpha+\sigma)t} \; \forall \; t \geq 0$ and $\|V(t)\| \leq Ke^{\sigma t} \; \forall \; t \leq 0$ for some constants $K, \alpha, \sigma > 0$. (By the HW exercise.) Thus

$$\|y^{(j+1)}(t,a) - y^{(j)}(t,a)\| \leq K \int_0^t e^{-(\alpha+\sigma)(t-s)} \|G(y^{(j)}(s,a)) - G(y^{(j-1)}(s,a))\| ds \\ + \int_t^\infty Ke^{\sigma(t-s)} \|G(y^{(j)}(s,a)) - G(y^{(j-1)}(s,a))\| ds. \tag{2}$$

This is one more reason to have the limits from t to ∞ in the second integral. We assume inductively that $\|y^{(j)} - y^{(j-1)}\| \leq C_j e^{-\mu t}$ for appropriately chosen C_j and μ. Upon substitution, we see that the induction step can be completed (for instance) if we assume that $C_j = \frac{C}{2^{j-1}}$ (for some C to be chosen later) and $\mu = \alpha$ if $\epsilon < \frac{\sigma}{4K}$ and $\|y^{(j)}\| < \delta$ for all j. Since the induction hypothesis must be met for $j = 1$, we see that $C = K\|a\|$ works. The condition involving δ can be met (inductively) if $\|a\| < \frac{\delta}{2K}$. We see that we get a Cauchy sequence that converges uniformly in t (for all $t \geq 0$) to a continuous function y. By uniform convergence (how), we see that the integral equation is met for all t and hence by the fundamental theorem of calculus, y is C^1. It also satisfies $\|y\| \leq 2K\|a\|e^{-\sigma t}$ for $\|a\| < \delta' = \frac{\delta}{2K}$.

From the integral equation and the initial data, we see that the ODE is satisfied. Moreover, $y_i(t) = -\int_t^\infty V(t-s)G(y(s, a_1, \ldots, a_k, 0, \ldots))ds$ for $i \geq k+1$. The graph S given by $(a_1, \ldots, a_k, \psi(a_1, \ldots, a_k) = -\int_0^\infty V(-s)G(y(s, a_1, \ldots, a_k, 0, \ldots))ds)$ turns out to be C^1. Indeed, it is not hard to prove that y depends on $a_1, \ldots$ in a C^0 manner by using the successive approximations and induction. To prove that it is C^1 requires more work (and the idea is similar to the proof of differentiable dependence on parameters). One can also show that S is tangent to the stable subspace at the origin.

For $\|a\| < \delta'$, we show that a solution of the integral equation above satisfying $\|y\| < \delta'$ is unique and hence the one we constructed is *the* solution: Suppose $\tilde{y}$ is another such solution. Then $\|\tilde{y} - y\| \leq 2\frac{K}{\sigma}\epsilon M$ where M is the supremum of $\|\tilde{y} - y\|$. Thus we have a contradiction unless $M = 0$.

We see using the estimates that as $t \to \infty$, $y \to 0$ if the initial data lies on S. We need to now prove that if we start with a point close to the origin and not on S, then $\|y\|$ cannot be $\leq \delta$ for all $t \geq 0$. This also proves that S is invariant (why?). Indeed we now prove that if $y(0) \notin S$ and is small, then $\|y(t)\| \leq \delta$ cannot be met for all $t \geq 0$: Simply use the Duhamel formula to get a contradiction as $t \to \infty$.