1 Recap

1. Observed that linear ODE can be made into first order systems.

2. Proposed two strategies to solve them. Successfully showed that if A is diagonalisable, then the space of real solutions is of dimension n (which is also the complex dimension of the space of complex solutions).

2 Linear systems of ODE

Here is a way to solve the damped oscillator: $y'' = -ky - by'$ where y is real-valued. Write $y' = v$, $v' = -bv - ky$. Then

$$\begin{bmatrix} y' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k & -b \end{bmatrix} \begin{bmatrix} y \\ v \end{bmatrix}. \tag{1}$$

The eigenvalues of A are $\frac{-b \pm \sqrt{b^2-4k}}{2}$. There are a few possibilities:

1. $b^2 \neq 4k$: In this case we have distinct (possibly complex) eigenvectors. Suppose we choose the columns of P^{-1} to be a basis of eigenvectors. Then $y = Ae^{\lambda t} + Be^{\bar{\lambda} t}$. Since y is real, $A = \bar{B}$ (why?) and thus $y = Ae^{\lambda t} + \bar{A}e^{\bar{\lambda} t} = e^{-bt}(a\cos(\sqrt{4k-b^2}t) + b\sin(\sqrt{4k-b^2}t))$.

2. $b^2 = 4k$: In this case the matrix has equal eigenvalues (equal to b) and is not diagonalisable! (why?) One eigenvector is $\vec{w} = \begin{bmatrix} 1 \\ -\frac{b}{2} \end{bmatrix}$. Let $P^{-1} = \begin{bmatrix} 1 & 0 \\ -\frac{b}{2} & 1 \end{bmatrix}$. Note that $PAP^{-1} = \begin{bmatrix} -\frac{b}{2} & 1 \\ 0 & -\frac{b}{2} \end{bmatrix}$. Thus $z_2 = (z_2)_0 e^{-bt/2}$ and $z_1' = -\frac{b}{2}z_1 + z_2 = -\frac{b}{2}z_1 + (z_2)_0 e^{-bt/2}$. Thus $(z_1 e^{b/2t})' = (z_2)_0$ and hence $z_1 = (z_2)_0 te^{-b/2t} + (z_1)_0 e^{-b/2t}$.

More generally, one can prove that

Theorem 1. *There* always *exists a matrix P so that $PAP^{-1} = J$ is in the Jordan canonical form, i.e., a block diagonal matrix where each block is of the form $\lambda I + N$ where $N = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}$ and λ is an eigenvalue. Upto permutation, this form is unique.*

Here is a very brief sketch of the ideas in the proof:
Suppose J is a Jordan block. How does one recover the basis vectors e_i from this information? Now $ker(J - \lambda I)$ is 1-dimensional and generated by the eigenvector e_1. Now $Ne_2 = e_1$, $Ne_3 = e_2$ and so on. In other words, if we know e_n, then $Ne_n, N^2e_n, \ldots,$ give us all the other basis vectors. Now take the preimage of e_i (inductively). It will be of the form $e_{i+1} + c_i e_1$ for some c_i. Choose some elements $\tilde{e}_i$ in the preimages. Now we choose $e_2 = N(\tilde{e}_3)$, $e_3 = N(\tilde{e}_4)$ and so on. (The last one is not uniquely determined.) Moreover, if A is in the Jordan form, then $Ker((A-\lambda I)^k) = Ker((A-\lambda I)^{k+1})$ for some

k. Note that this space is taken to itself by A. It suffices to determine the number of Jordan blocks for λ of each size. Suppose b_k is the number of such blocks of size at least k, then $b_k = dim(ker(A-\lambda I)^k)) - dim(ker(A-\lambda I)^{k-1})$ (why?). Thus we can determine the JCF uniquely.
We turn this strategy around for existence. Actually, we will simply provide an idea of the algorithm in a special case here: Consider the space $Ran(A-\lambda I)$ for an eigenvalue λ. It has dimension strictly less than n. Consider the largest k such that $Ran(A-\lambda I)^k = Ran(A-\lambda I)^{k+1}$. Now take $U = ker(A-\lambda I)^k$. Note that A takes U to itself. So we can assume WLOG that A has only one eigenvalue λ. By considering $A-\lambda I$, assume that $\lambda = 0$. Also assume that there are only 1×1 and 2×2 Jordan blocks. Note that $ker(A)$ consists of eigenvectors. Take $ker(A^2) \cap Ran(A)$. It ought to consist of eigenvectors and 2×2 Jordan blocks. Take the preimages of the eigenvectors of A lying in $ker(A^2)$. They generate the 2×2 Jordan blocks and so on. $\square$
Assuming this theorem, we can iteratively solve each Jordan block to get $\vec{z}$ as a function involving polynomials and exponentials. Indeed, suppose we let A be a Jordan block of size n. Then $z_n' = \lambda z_n$, $z_{n-1}' = z_n + \lambda z_{n-1}$ and so on. Thus $z_n = (z_n)_0 e^{\lambda t}$, $(z_{n-1}e^{-\lambda t})' = (z_n)_0$ and hence $z_{n-1} = (z_n)_0 t e^{\lambda t} + (z_{n-1})_0 e^{\lambda t}$. Now $z_{n-i}' = z_{n-i+1} + \lambda z_{n-i}$ and hence $z_{n-i} = (z_{n-i})_0 e^{\lambda t} + e^{\lambda t}\int_0^t e^{-\lambda s} z_{n-i+1}(s) ds$. Inductively, we see that (HW) $\vec{z}$ is a (complex) linear combination of n linearly independent vector-valued functions. Thus, we can show that if A is an $n \times n$ matrix, the (real and complex) dimension of the space of solutions is n.
Moreover, if $y(0) = 0$, then $y(t)$ is identically zero (from the formulae). Therefore (why?) the solution is unique. This also implies that if $y(0)$ is real, then so is $y(t)$ (why?)