# 1 Recap

1. Stated the theorem of the Jordan canonical form and illustrated with a simple $2 \times 2$ example.

2. Proved that $y' = Ay$ has an $n$-dimensional space of solutions.

# 2 Linear systems of ODE

## 2.1 Exponentiation

We shall define a notion of the matrix exponential $e^A$ for a square matrix $A$. To this end, we recall the definition of norm $\|.\| : V \to \mathbb{R}$ on a real/complex vector space $V$:

1. $\|v\| \geq 0$ with equality iff $v = 0$.

2. $\|av\| = |a|\|v\|$.

3. $\|v + w\| \leq \|v\| + \|w\|$.

An obvious example of a norm is one that comes from an inner product: $\|v\|^2 = \langle v, v\rangle$. However, not all norms arise this way (for instance, the "taxi-cab" norm on $\mathbb{R}^n$: $\|v\| = \sum_i |v_i|$). If indeed a norm arises from an inner product, it is easy to see that the polarisation identity holds: $\|v+w\|^2 + \|v-w\|^2 = 2\|v\|^2 + 2\|w\|^2$. If this identity holds, then indeed $\langle v, w\rangle := \frac{\|v+w\|^2 - \|v-w\|^2}{2}$ can be proven to be an inner product.

Two norms are said to be equivalent if there exists a constant $C > 0$ such that $\frac{1}{C}\|v\|_1 \leq \|v\|_2 \leq C\|v\|_1$ for all vectors $v$. One can prove that on a finite-dimensional vector space, all norms are equivalent to each other.

On the space of $n \times n$ real/complex matrices there are natural norms. One is the Hilbert-Schmidt norm: $\|A\|_{HS}^2 = \sum_i |A_{ij}|^2$. Another is the operator norm $\|A\|_{op} = \max_{|v|=1} \|Av\|$. These norms are sub-multiplicative, i.e., $\|AB\| \leq \|A\|\|B\|$ (why?)

Given these notions, on a normed vector space, we can talk about convergence of sequences: $a_n \to a$ if $\|a_n - a\| \to 0$ in the sense of real numbers. We have the usual properties of convergence. In addition, on $\mathbb{R}^n$ or $\mathbb{C}^n$ (or for that matter, any finite-dimensional vector space with a basis), $a_n \to a$ iff the individual components converge (why?). We now make the following definition.

Definition: Let $A$ be a complex/real $n \times n$ matrix. Define $e^A := I + A + A^2/2! + \ldots$.

Lemma: This series converges.

Proof: We shall prove that this series is Cauchy. Then each entry forms a Cauchy sequence (why?) and hence by the completeness of reals, it converges. Indeed, $\|\sum_{k=n}^m \frac{A^k}{k!}\| \leq \sum_{k=n}^m \frac{\|A^k\|}{k!} \leq \sum_{k=n}^m \frac{\|A\|^k}{k!}$ (why?) Now we know that $e^{\|A\|}$ exists as a convergent series (why?) and hence we are done (why?) $\square$

We have a couple of properties (most are easy to prove):

1. Suppose $B = PAP^{-1}$. Then $e^B = Pe^AP^{-1}$.

2. If $A$ is block diagonal with diagonal entries $A_i$, then so is $e^A$ with diagonal entries $e^{A_i}$.

3. If $AB = BA$, then $(A+B)^k = \sum \binom{n}{k} A^k B^{n-k}$ and $e^{A+B} = e^A e^B$ . (In general, this is not true. In fact, the quest to find a relationship between $e^{A+B}$ and $e^A e^B$ leads to the theory of Lie algebras. (The Baker-Campbell-Hausdorff formula.)

4. $e^A$ is invertible and $(e^A)^{-1} = e^{-A}$.

5. $\|e^A\| \leq e^{\|A\|}$.

6. If $J = \lambda I + N$ is a Jordan block, then $e^J = e^{\lambda} e^N = e^{\lambda}(I + N + N^2/2! + \ldots + N^{n-1}/(n-1)!$ because $N^n = 0$. As a consequence, $\|e^{tA}| \leq ke^{-rt}$ for all $t \geq 0$ if the real parts of all eigenvalues of $A$ are strictly negative. (HW)

An interesting point: If we guess at $2$ linearly independent solutions to $y'' = -k^2 y$, we are done because of the theorems above. We can try $\sin(kx), \cos(kx)$ and they work. This raises a question: Suppose $u_1, u_2$ are two differentiable functions, when can we say that they are linearly independent? Indeed, suppose $c_1 u_1 + c_2 u_2 = 0$, then $c_1 u_1' + c_2 u_2' = 0$. Thus this system does not have a non-trivial solution if $\det \begin{pmatrix} u_1 & u_2 \\ u_1' & u_2' \end{pmatrix} \neq 0$. More generally, $u_1, \ldots, u_n$ are linearly independent if a similar determinant involving higher derivatives does not vanish (what determinant?) This determinant is called the Wronskian.
Another interesting application of the Wronskian: If $y'' = a(x)y' + b(x)y$, then $W(x)$ the Wronskian, satisfies $W' = aW$ and hence $W$ is known. (As a consequence, if the Wronskian vanishes at one point, it does so everywhere.) Now we can solve for one of the linearly independent solutions knowing the other!