1 Recap

1. Defined the Wronskian and proved that is non-vanishing can be used to prove linear independence. Moreover, used it to guess another solution of a homogeneous second order linear equation if one is already known.

2. Defined the matrix exponential and proved some properties (including the ability to calculate it using Jordan blocks).

2 Linear systems of ODE

We have one last property.

1. $\det(e^A) = e^{tr(A)}$: $\det(e^A) = \det(Pe^J P^{-1}) = \det(e^J) = e^{\sum_i \lambda_i} \det(e^{N_1}) \det(e^{N_2}) \ldots$. Now e^N is upper-triangular with diagonal entries equal to 0 (why?) and thus $\det(e^N) = 1$. Thus we are done.

Now we define a matrix-valued function of 1-variable to be continuous/differentiable iff each entry is so. We claim that if $A(t), B(t)$ are differentiable then so is AB and $(AB)' = A'B + AB'$ (exercise). Moreover,
Lemma: e^{tA} is differentiable and its derivative is Ae^{tA}.
Proof: $e^{tA} = Pe^{tJ}P^{-1}$. Now for a pure Jordan block J_i, $e^{tJ_i} = e^{t\lambda}e^{tN_i}$. Now e^{tN_i} has polynomial entries in t and is hence differentiable. Thus e^{tA} is differentiable. Now $e^{tN_i} = I + tN_i + t^2N_i^2/2! + \ldots + t^{k-1}N_i^{k-1}/(k-1)!$ and hence $(e^{tN_i})' = N_i(1 + tN_i + \ldots) = N_i e^{tN_i}$. Hence $(e^{tJ_i})' = J_i e^{tJ_i}$ and $(e^{tA})' = Ae^{tA}$.
Now we can solve $y' = Ay$: Note that $(e^{-At}y)' = e^{-At}y' - e^{-At}Ay = 0$ and hence $e^{-At}y = y_0$ and $y = e^{At}y_0$. Here is a basis of solutions: $e^{At}e_i$. The space of solutions is n-dimensional. We can calculate the matrix exponential using the Jordan canonical form (or any other method of our choice).
We can also solve the inhomogeneous autonomous problem $y' = Ay + f$: $e^{-At}(y' - Ay) = e^{-At}f$ and hence $(ye^{-At})' = e^{-At}f$ and $y = e^{At}y_0 + e^{At}\int_0^t e^{-As}f(s)ds$ or more generally, $y = e^{A(t-t_0)}y(t_0) + e^{A(t-t_0)}\int_{t_0}^t e^{-As}f(s)ds$. This formula is called the Duhamel formula/method of variation of parameters.
Here is an example: $y'' = -ky - by' + f$. We already know what e^{At} is in this case (why?) Now we have reduced the problem to an integral involving f. If f is exponential, the integral is easy. Otherwise, one cannot do much more.
In older parlance, if we know one solution (a "particular" solution) to $y' = Ay + f$, then every solution is $y = y_0 + h$ where h is a solution of the homogeneous problem (given by $e^{At}y_0$). Since the particular solution is obtained by simply making y_0 into a specific function (as opposed to a constant), the name "variation of parameters" came about.
We shall now discuss only uniqueness (existence will be considered later) for a non-autonomous system $y' = A(t)y + B(t)$ where $A(t), B(t)$ are continuous functions on $[a, b]$ with $y(t_0) = y_0$ where $t_0 \in [a, b]$. Without loss of generality, $t_0 = 0$ (why?). Suppose u, v are differentiable solutions with $u(t_0) = v(t_0)$, then $(u - v)' = A(t)(u - v)$. Thus $u - v = \displaystyle\int_{t_0}^t A(u - v)ds$. Thus $\|u - v\| = \| \displaystyle\int_{t_0}^t A(u - v)ds$. Now we have the following lemma.

Lemma 2.1. *Let* $f : [a,b] \to \mathbb{R}^n$ *be a continuous function. Then* $\| \int_{t_0}^{t} f(s)ds \| \leq \int_{t_0}^{t} \|f(s)\| ds$.

Proof. Indeed, let $v(t) = \int_{t_0}^{t} f(s)ds$. Then $\|v\|^2 = v. \int_{t_0}^{t} f(s)ds = \int_{t_0}^{t} v(t).f(s)ds \leq \int_{t_0}^{t} \|v(t)\| \|f(s)\| ds = \|v(t)\| \int_{t_0}^{t} \|f(s)\| ds$. Hence we are done. □

Thus, $\|u - v\| \leq \int_{t_0}^{t} \|A(t)\| \|u - v\|$. If $A(t)$ is continuous, then so is $\|A(t)\|$ and thus $\|A(t)\| \leq C$ on $[a,b]$. Using Gronwall's inequality (HW problem), we see that $\|u - v\| \leq e^{C(t-t_0)} \|u - v\|(t_0) = 0$. Hence $u = v$. □