1 Recap

1. Examples and counterexamples for real-analyticity.
2. Characterisation of real-analytic functions
3. Solving the (approximation of the) simple pendulum using power series.

2 Real-analytic functions

1. $y'' - 2yt' + 2py = 0$ where $p \in \mathbb{R}$ (the Hermite equation): Again, if y is twice differentiable, then from the equation it is smooth (why?). We know that if there is a solution it is unique given y, y' at $t = 0$. Now we try to find it using power series. $y = \sum a_n t^n$. It satisfies $(n+1)(n+2)a_{n+2} = -2(p-n)a_n$. Thus $y = a_0 y_1 + a_1 y_2$. It is easy to check that y_1, y_2 are convergent on $\mathbb{R}$, and that they are linearly independent. If $p = 0, 1, 2, 3, \ldots$ one of these solutions becomes a polynomial. (These polynomials are called Hermite polynomials and satisfy $H_n = (-1)^n e^{t^2} \frac{d^n(e^{-t^2})}{dt^n}$.) These polynomials also occur in quantum mechanics as the eigenstates (after multiplying by a Gaussian) of the harmonic oscillator. They are also used in numerical integration (Gauss-Hermite quadrature).

We now prove a general result.

Theorem 1. *Consider $\vec{y}' = A(t)\vec{y}(t) + B(t)$ where A, B are real analytic matrix-valued functions at $t = t_0$ such that they are power series on $(t_0 - R, t_0 + R)$. Then there exists a unique solution to this equation with $\vec{y}(t_0), \vec{y}'(t_0)$ specified. Moreover, the solution is real-analytic at $t = t_0$ with radius of convergence at least R.*

Proof. Uniqueness: We have already proven uniqueness of differentiable solutions given $y(t_0)$ using the Gronwall inequality approach. By the way, any differentiable solution is smooth on $(t_0 - R, t_0 + R)$ (why?).
Existence: Without loss of generality, assume that $t_0 = 0$. The radius of convergence of A and B is at least R. Let $r < R$ be fixed but arbitrary. The power series $A(t) = \sum A_n t^n$ and $B(t) = \sum \vec{b}_n t^n$ converge uniformly on $[-r-\epsilon, r+\epsilon]$. We shall try to produce a real-analytic solution that uniformly converges on $[-r, r]$ as $\vec{y}(t) = \sum \vec{c}_n t^n$. Then

$$(n+1)\vec{c}_{n+1} = \sum_{k=0}^{n} A_k \vec{c}_{n-k} + \vec{b}_n \;\forall\, n \geq 0. \tag{1}$$

Inductively, we can solve for $\vec{c}_n$ uniquely given $\vec{c}_0$. The question is whether we get a convergent power series. We see that

$$(n+1)\|\vec{c}_{n+1}\| \leq \sum_{k=0}^{n} \|A_k\|\|\vec{c}_{n-k}\| + \|\vec{b}_n\|. \tag{2}$$

Since A, B are uniformly convergent power series, $\|A_k\| \leq \frac{M}{(r+\epsilon)^k}$ and $\|\vec{b}_n\| \leq \frac{M}{(r+\epsilon)^k}$. Let $\tilde{r} = r + \epsilon$. Let u_n satisfy the following identities.

$$(n+1)u_{n+1} = \sum_{k=0}^{n} \frac{M}{\tilde{r}^k} u_{n-k} + \frac{M}{\tilde{r}^n}. \tag{3}$$

If $u_0 = \|\vec{c}_0\|$, then $u_n \geq \|\vec{c}_n\| \; \forall \; n \geq 0$ (why?). Writing the same as above for n,

$$nu_n = \sum_{k=0}^{n-1} \frac{M}{\tilde{r}^k} u_{n-1-k} + \frac{M}{\tilde{r}^{n-1}}. \tag{4}$$

Thus

$$\begin{aligned} (n+1)u_{n+1} &= Mu_n + \frac{nu_n}{\tilde{r}} \\ \Rightarrow \frac{u_{n+1}}{u_n} &= \frac{M\tilde{r} + n}{(n+1)\tilde{r}} \end{aligned} \tag{5}$$

The limit is $\frac{1}{\tilde{r}}$. Thus we are done (why?). □