1 Recap

1. Stated and proved a theorem about real-analytic non-autonomous linear systems having real-analytic solutions.

2 Real-analytic functions

While the theorem above is pleasant, it does not cover all cases of interest. For example, suppose we want to study an electrostatic field in a long cylinder with potential (cylindrical symmetry) specified on the cylinder, we will have to solve the Laplace equation in cylindrical coordinates (r, θ, z). That is,

$$\frac{1}{r}\partial_r(r\phi_r) + \frac{1}{r^2}\phi_{\theta\theta} + \phi_{zz} = 0. \tag{1}$$

We use the method of separation of variables, i.e., $\phi = R(r)P(\theta)Z(z)$. Then we see that R satisfies

$$\frac{r}{R}R' + \frac{r^2}{R}R'' - \lambda r^2 = constant. \tag{2}$$

That is, it is of the form (after changing variables and solving the other equations) $y'' + \frac{y'}{t} + y\left(1 - \frac{\nu^2 s}{t^2}\right) = 0$ where ν is an integer. This is an example of Bessel's equation. (The solutions are called Bessel functions $J_\nu(t)$.) It occurs in various other situations in real life (and in probability I believe as the pdf of a product of two normal variables). Likewise, if we try to solve the Laplace in spherical coordinates (or more generally, the eigenvalues problem arising from the Hydrogen atom for instance), after separation of variables, the solutions to the angular part are $P_n(\cos(\theta))$ where P_n are the Legendre polynomials satisfying the equation

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0. \tag{3}$$

Suppose we declare $v = y'$, these equations do not fall under the purview of the theorem above because A, B are no longer real-analytic. They have singularities (why?) In fact, if we try $v = ty'$, then the equations fall under the purview of

$$\vec{y}' = \frac{A(t)}{t}\vec{y}, \tag{4}$$

where $A(t)$ is real-analytic. Such systems are called systems with *regular singular points* (an oxymoron?) Suppose $\vec{y}(t_0) = \vec{y}_0$ where $t_0 > 0$ (the case of < 0 is similar), then declare $s = \ln(t)$. We can now consider

$$\frac{d}{ds}\vec{y}(e^s) = A(e^s)\vec{y}(e^s). \tag{5}$$

The usual Frobenius method above can be applied to this equation to conclude the existence of a real-analytic solution in a neighbourhood of $s_0 = \ln(t_0)$. Changing variables, we see that y is a power series in $\ln(t)$. Now here is an interesting lemma (proof is a HW exercise).

Lemma 2.1. *Let $f(t)$ be real-analytic at $g(t_0)$ and $g(t)$ at t_0. Then $h(t) = f(g(t))$ is real-analytic at t_0.*

Thus y is real-analytic in t near t_0. However, what can we say about the power series really? For instance, if $A(t)$ is a constant, then in terms of s, $\vec{y} = e^{A(s-s_0)}\vec{y}_0$. Thus, $\vec{y}(t) = e^{A\ln(t)}e^{-A\ln(t_0)}\vec{y}_0$. Now writing A in the Jordan canonical form (and noting that the ratio test and so on apply to complex power series as well), $e^{A\ln t} = Pe^{J\ln(t)}P^{-1}$. If J is a Jordan block, then $e^{J\ln(t)} = t^{\lambda_R}(\cos(\lambda_I \ln(t)) + \sqrt{-1}\sin(\lambda_I \ln(t)))(I + N\ln(t) + N^2(\ln(t))^2/2! + \ldots)$. Thus we can in general hope for a solution involving linear combinations of functions of the type $t^r \sum a_n t^n (p(\ln(t))$ where r and a_n are in general, complex, and p is a possibly complex polynomial.
Let us specialise to equations of the form $y'' + P(t)y' + Q(t)y = 0$. If we want this to have regular singular points at $t = 0$, then $P(t)t$ and $Q(t)t^2$ are real-analytic around 0 (why?) Thus $P(t) = \frac{p_0}{t} + p_1 + p_2 t + \ldots$, $Q(t) = \frac{q_0}{t^2} + \frac{q_1}{t} + q_2 + \ldots$. Let $y(t) = t^m(a_0 + a_1 t + \ldots)$. Then $y' = \sum_{n=0} a_n(m+n)t^{m+n-1}$, $y'' = t^{m-2}\sum a_n(m+n)(m+n-1)t^n$. Now $P(t)y' = \frac{1}{t}\sum p_n t^n \sum a_k(m+k)t^{m+k-1}$ and so on. Thus

$$a_n f(m+n) + \sum_{k=0}^{n-1} a_k((m+k)p_{n-k} + q_{n-k}) = 0, \tag{6}$$

where $f(m+n) = (m+n)(m+n-1) + (m+n)p_0 + q_0 \ \forall \ n \geq 0$. For $n = 0$ we see that $f(m) = 0$.