HW 1 (Quiz in on Jan 14)

1. Suppose $z_{n-i} = (z_{n-i})_0 e^{\lambda t} + e^{\lambda t} \int_0^t e^{-\lambda s} z_{n-i+1}(s) ds$ and $z_n(t) = e^{\lambda t} z_n(0)$. Then prove that $\vec{z}$ is a complex linear combination of n complex linearly independent vector-valued functions, each of which solves $\vec{z}' = J\vec{z}$. Now conclude that indeed the space of real solutions of $\vec{y}' = A\vec{y}$ is n real dimensional. Also prove that solutions are unique.

2. (Problem 14 in Chapter 3 in Nandakumaran's book):

   (a) Consider $y' + py = q$ where p, q are continuous functions. Show that if $q \geq 0$, then $y \geq 0$ if $y(0) \geq 0$.

   (b) Consider $x' + p_1 x = q$ and $y' + p_2 y = q$. Show that if $p_2 \geq p_1$, $x(0) \geq y(0)$ and $y \geq 0$, then $x \geq y$.

   (c) Consider $y' + py \leq q$. Derive the inequality

   $$y(t) \leq \exp\left(-\int_0^t p(s)ds\right)\left[y(0) + \int_0^t q(s)\exp\left(\int_0^s p(z)dz\right)\right].$$

   (d) Derive Gronwall's inequality: Assume that f, g are continuous on $[a, b]$ and $g \geq 0$. Also assume that $f(t) \leq c + k\displaystyle\int_{t_0}^t f(s)g(s)ds$. Then prove that $f(t) \leq c\exp\left(k\int_{t_0}^t g(s)ds\right)$.

3. Solve the following equation (using the Jordan canonical form): $y' = Ay$ where $A = \begin{bmatrix} 5 & 4 & 2 & 1 \\ 0 & 1 & -1 & -1 \\ -1 & -1 & 3 & 0 \\ 1 & 1 & -1 & 2 \end{bmatrix}$