HW 2

1. If A, B are matrix-valued differentiable functions such that AB makes sense as a matrix, prove that AB is differentiable and that $(AB)' = A'B + AB'$.

2. Assume existence of a solution to $y^{(n)} + p_1(t)y^{(n-1)} + \ldots + p_n(t)y = 0$ on $(-\epsilon, \epsilon)$ with $y(0) = y_0, y'(0) = y_1, \ldots, y^{(n-1)}(0) = y_{n-1}$ (where p_i are continuous functions). Prove that the set of real solutions is an n-dimensional subspace of the space of all n-times differentiable functions.

3. (Slightly tricky) For the previous n^{th} order differential equation, let $u_1, \ldots, u_n$ be n solutions. Define the Wronskian W of these solutions. By differentiating it, prove that if W vanishes somewhere, it vanishes everywhere. Now prove that W vanishes somewhere (and hence everywhere) iff $u_1, \ldots, u_n$ are linearly dependent (note that this property is not true in general but for solutions of ODE, it is so!)

4. Let A be a complex square matrix such that for all eigenvalues of A, their real parts are strictly negative. Then prove that there exist strictly positive constants k, r such that $\|e^{At}\| \leq ke^{-rt} \; \forall \; t \geq 0$.