1 Recap

1. Set up the stage for Frobenius' method with the indicial equation $a_n f(m+n) + \sum_{k=0}^{n-1} a_k((m+k)p_{n-k} + q_{n-k}) = 0$ where $f(m+n) = (m+n)(m+n-1) + (m+n)p_0 + q_0 \;\forall\; n \geq 0$. (Note that $a_0 f(m) = 0$ and we want a_0 to be a free parameter. So we want $f(m) = 0$.)

2 Real-analytic functions

It turns out (Frobenius' theorem) that $\sum a_n t^n$ and $\sum b_n t^n$ converge on $(-R, R)$ if $\sum p_n t^n$ and $\sum q_n t^n$ do. Here is the proof: First fix $r < R$. We see that $|p_k| + |q_k| \leq \frac{C}{r^k}$.

1. $f(m_2 + n) \neq 0$ (In particular, if $m_1 - m_2$ is not a non-negative integer):

$$|a_n| \leq \frac{C}{n^2} \sum_{k=0}^{n-1} |a_k| \frac{1}{r^{n-k}} \left((m+k)+1\right)$$
$$\Rightarrow |a_n| \leq \frac{C}{n} \sum_{k=0}^{n-1} |a_k| \frac{1}{r^{n-k}}. \tag{1}$$

Again, we define u_n with $u_0 = |a_0|$ satisfying

$$u_n = \frac{C}{n} \sum_{k=0}^{n-1} \frac{u_k}{r^{n-k}}. \tag{2}$$

We see that $u_n \geq |a_n|$ by induction. Thus $u_n = u_{n-1}\left((n-1)/nr + C/n\right)$. Thus $\lim \frac{u_n}{u_{n-1}} = \frac{1}{r}$. Thus $\sum u_n (r-\epsilon)^n$ converges and by the Weierstrass-M test, $\sum a_n t^n$ converges uniformly on $[r-\epsilon, r+\epsilon]$ and since r, ϵ are arbitrary, we are done. We claim that if $m_1 - m_2$ is not an integer, then these two solutions are linearly independent (HW).

2. $m_1 = m_2 + n_0$ where n_0 is a non-negative integer: The above argument works for m_1. Now we try $y_2 = t^{m_2} \sum b_n t^n + C \ln(t) y_1$. Substituting into the ODE we get (after simplification using the fact that y_1 is a solution)

$$(t^{m_2} \sum b_n t^n)'' + P(t)(t^{m_2} \sum b_n t^n)' + Q(t)(t^{m_2} \sum b_n t^n) - \frac{C y_1}{t^2} + \frac{2C y_1'}{t} + \frac{C P y_1}{t} = 0. \tag{3}$$

$$a_n f(m+n) + \sum_{k=0}^{n-1} a_k((m+k)p_{n-k} + q_{n-k}) = 0, \tag{4}$$

where $f(m+n) = (m+n)(m+n-1) + (m+n)p_0 + q_0 \;\forall\; n \geq 0$. For $n = 0$ we see that $f(m) = 0$. Now we have two cases.

1. $m_1 = m_2 + n_0$ where $n_0 \geq 1$: The above argument works for m_1 to produce a solution y_1. For b_n, we see that

$$(t^{m_2} \sum b_n t^n)'' + P(t)(t^{m_2} \sum b_n t^n)' + Q(t)(t^{m_2} \sum b_n t^n) - \frac{Cy_1}{t^2} + \frac{2Cy_1'}{t} + \frac{CPy_1}{t} = 0. \tag{5}$$

Thus defining $a_k = 0 = b_k$ for negative k,

$$C\left((2(m_1 + n - n_0) - 1)a_{n-n_0} + \sum_{k=0}^{n-n_0} p_k a_{n-n_0-k}\right)$$
$$+ b_n f(m_2 + n) + \sum_{k=0}^{n-1} b_k((m_2 + k)p_{n-k} + q_{n-k}) = 0. \tag{6}$$

For $n = n_0$ we see that

$$C(2m_1 - 1 + p_0)a_0 + \sum_{k=0}^{n_0-1} b_k((m_2 + k)p_{n_0-k} + q_{n_0-k}) = 0$$

and hence we can solve for C (because inductively, we can solve for b_k up to $n_0 - 1$). By the way, $2m_1 - 1 + p_0 \neq 0$ because by assumption the quadratic $f(x)$ does NOT have multiple roots and hence $f'(m_1) = 2m_1 - 1 + p_0 \neq 0$. Now $f(m + n) \neq 0$ for $n > n_0$. Thus we can solve for all the other b_k. We now have to prove that $\sum_n b_n t^n$ converges absolutely and uniformly on $[-(r - \epsilon), r + \epsilon]$ where $r < R$ and $\epsilon > 0$ are arbitrary. Note that $|a_k| \leq \frac{C}{r^k}$ for all $k \geq 0$. Hence for all sufficiently large n ($\geq N$) we see that

$$|b_n| \leq \frac{C}{n^2} \sum_{k=0}^{n-1} \frac{|b_k|(m_2 + k + 1)}{r^{n-k}} + \frac{Cn}{r^n}. \tag{7}$$

As usual, we define $u_N = |b_N| > 0$ (without loss of generality) and u_n to satisfy

$$u_n = \frac{C}{n^2} \sum_{k=0}^{n-1} \frac{u_k(m_2 + k + 1)}{r^{n-k}} + \frac{Cn}{r^n}. \tag{8}$$

Note that $u_n \geq |b_n| > 0$ inductively and that $u_n \geq \frac{Cn}{r^n}\ \forall\, n \geq N + 1$. Now we see that

$$u_n = u_{n-1}\left(\frac{n-1}{nr} + \frac{C}{n}\right) + \frac{Cn}{r^n} - \frac{C(n-1)}{r^{n-1}}\left(\frac{n-1}{nr} + \frac{C}{n}\right)$$
$$\Rightarrow \frac{u_n}{u_{n-1}} = \left(\frac{n-1}{nr} + \frac{C}{n}\right) + \frac{C}{r^n u_{n-1}}\left(2 + \frac{1}{n} - \frac{C(n-1)r}{n}\right)$$
$$\Rightarrow \lim \frac{u_n}{u_{n-1}} = \frac{1}{r}. \tag{9}$$

To be continued....