1 Recap

1. Proved most of Frobenius. Produced y_1, y_2 in the case where $n_0 \geq 1$ is an integer.

2 Frobenius' theorem

1. $m_1 = m_2 + n_0$ where $n_0 \geq 1$: At this point we have two solutions $y_1, y_2 = t^{m_2} \sum b_n t^n + C \ln(t) y_1$. However, it seems that there are three arbitrary constants, a_0, b_0, b_{n_0}. However, if we take a general solution $\alpha y_1 + \beta y_2$, firstly, absorb the α into a_0, and β into b_0 to assume that $\alpha = \beta = 1$ Wlog. Now $y_1 + y_2 = t^{m_1} \sum a_n t^n + C \ln(t) y_1 + t^{m_1} b_{n_0} + t^{m_2} \sum_{n \neq n_0} b_n t^n$. Now note that b_{n_0} can be absorbed into a_0 as well! Thus WLog $b_{n_0} = 0$. To prove that these solutions are linearly independent is a HW exercise. □

2. $m_1 = m_2$: In this case we try $y_2 = \ln(t) y_1 + t^m \sum_{n=1}^{\infty} b_n t^n$. The calculations are similar and left as an exercise.

3 General existence and uniqueness theory

We have seen several examples of things that can go wrong in ODE (two solutions (in fact infinitely many (why?)) and a solution that blows up in finite-time). Here are examples where you have no solutions at all

1. $ty' = y$ with $y(0) = 2$. Note that the right-hand-side is 2 at 0 and the left-hand-side is 0. But you may object that I am cheating because it has a singularity.

2. $y' = -1$ when $y \geq 0$ and $y' = 1$ when $y < 0$. The right-hand-side is bounded (not continuous though). Let $y(0) = 0$. Then if there was a differentiable solution, it would have been decreasing at $t = 0$ and near $t = 0$. Thus $y(t) < 0$ for some $t > 0$. However, for all negative values of y, the solution is increasing (in fact, increasing at a constant rate). This is a contradiction!

So to solve $y' = f(y, t)$, it seems that f better be continuous. Since $y' = \sqrt{y}$ has no uniqueness, perhaps f being differentiable is even better. Going over our Gronwall inequality proof of uniqueness, it seems that the crucial point was $\|Ay\| \leq \|A\| \|y\|$. We can generalise this condition: Let $D \subset \mathbb{R}^n$ be an open connected set. A function $f : D \to \mathbb{R}^m$ is said to be locally Lipschitz if for every $x_0 \in D$, there exists a number M_{x_0} (called a Lipschitz constant) and a neighbourhood N_{x_0} of x_0 such that on N_{x_0}, $\|f(x) - f(y)\| \leq M_{x_0} \|x - y\|$. If the Lipschitz constant is independent of x, it is called Lipschitz.
Indeed, here is a general uniqueness result.

Theorem 1. *There exists at most one differentiable solution $y : [0, h) \to \mathbb{R}^n$ (where h is an extended real) to $y' = f(y, t)$ with $y(0) = y_0$ where $f : D \to \mathbb{R}^n$ is uniformly locally Lipschitz in y (that is, the Lipschitz constant for y is independent of t as well for a neighbourhood of the point) and D is a domain that contains $(y_0, 0)$.*

Proof. Suppose there exist two such solutions y_1, y_2. The set S of all $t \in [0, h)$ on which they coincide is non-empty ($0 \in S$) and closed (why?). If we just prove that it is open, we will be done (why?) So suppose $t_0 \in S$. We shall prove that an interval around t_0 is also contained in S. To be cont'd.... $\square$