1 Recap

1. Examples of things that can go wrong in general.
2. Locally Lipschitz.
3. Stated Uniqueness and uniformly locally Lipschitz. Reduced the problem to proving that a set $S \in [0, h)$ is open.

2 General existence and uniqueness theory

Proof. We shall prove that an interval around t_0 is also contained in S. Indeed, $(y_1 - y_2)' = f(y_1, t) - f(y_2, t)$ and hence $y_1 - y_2 = \int_{t_0}^{t} (f(y_1, s) - f(y_2, s))ds$. Thus $\|y_1 - y_2\| \leq |\int_{t_0}^{t} \|f(y_1, s) - f(y_2, s)\| ds|$. By the uniform local Lipschitzness of f, there exists an ϵ such that on $B_\epsilon(y(t_0)) \times (t_0 - \epsilon, t_0 + \epsilon)$, f is Lipschitz with constant M. Now since y_1, y_2 are continuous, on $(t_0 - \delta, t_0 + \delta)$ (where $\delta < \epsilon$) we have $y_1, y_2 \in B_\epsilon(y(t_0))$. Thus if $t \in (t_0 - \delta, t_0 + \delta)$, $\|y_1 - y_2\| \leq M |\int_{t_0}^{t} \|y_1 - y_2\| ds|$. Let $u = \|y_1 - y_2\|$ and assume that $t > t_0$ WLOg. Then $u \leq M \int_{t_0}^{t} u(s)ds$. By Gronwall's inequality, $u(t) = 0$ on $(t_0 - \delta, t_0 + \delta)$. $\square$

Now what are examples of functions that are uniformly locally Lipschitz in y?

1. $f(y, t) = e^y h(t)$ where h is continuous: Indeed, $|f(y_2, t) - f(y_1, t)| = e^{y_1} h(t) |e^{y_2 - y_1} - 1| \leq C |e^{y_2 - y_1} - 1|$ on a neighbourhood of y_1, t. Now $|e^h - 1| \leq C|h|$ for $|h| \leq 1$ for instance.
2. $f(y, t) = \sqrt{y}$ is NOT near $y = 0$: $\frac{\sqrt{y}}{y} = \frac{1}{\sqrt{y}} \to \infty$ as $y \to 0$.
3. $f(y, t) = |y| h(t)$ where h is continuous.
4. Here is a general example: Suppose $\frac{\partial f}{\partial y}$ exists on the domain and is continuous (jointly in t, y) then indeed, $|f(y_2, t) - f(y_1, t)| \leq \int_{y_1}^{y_2} |\frac{\partial f}{\partial y}| dy \leq C|y_2 - y_1|$ in a neighbourhood of (y_1, t).
5. So if f is compactly supported and smooth, it is Lipschitz (not just locally so).

Now we want to prove an existence result. How does one come up with something from nothing? Well, firstly, if were to solve this problem numerically, we would have tried Euler's method: $\Delta y \approx \Delta t f(y_0, t_0)$, if $y_1 = y_0 + \Delta y$, then $y_2 - y_1 \approx \Delta t f(y_1, t_1)$ and so on. Taking cue from this idea, let us try to solve $y(t) - y_0 = \int_{t_0}^{t} f(y(s), s)ds$ by an iterative method: That is, $y_{n+1}(t) = y_0 + \int_{t_0}^{t} f(y_n(s), s)ds$ with $y_0(t) = y_0$.

How can we hope to prove convergence? Well, we can try to look $y_{n+1} - y_n$ and if that decays quickly as $n \to \infty$, then y_n presumably converges. Thus, $y_{n+1}(t) - y_n(t) = \int_{t_0}^{t} (f(y_{n+1}(s), s) - f(y_n(s), s))ds$. This seems to be the right place to apply Gronwall and so on, provided we assume Lipschitzness.
Picard's theorem:

Theorem 1. *Let $D \subset \mathbb{R}^{n+1}$ be a domain and let $f : D \to \mathbb{R}^n$ be continuous (jointly) on D and Lipschitz in y on D with Lipschitz constant α. Let $R = \bar{B}_b(y_0) \times [t_0 - a, t_0 + a] \subset D$. Let $M = \max_R |f|$ and $h = \min\left(a, \frac{b}{M}, \frac{1}{2\alpha}\right)$. In the initial-value-problem (IVP) $y' = f(y,t)$ with $y(t_0) = y_0$ has a unique solution on $[t_0 - h, t_0 + h]$.*

Proof. Uniqueness was proven earlier. We shall prove existence by considering

$$y_{n+1}(t) = y_0 + \int_{t_0}^{t} f(y_n(s), s) ds \tag{1}$$

with $y_0(t) = y_0$. Suppose S_n is the set of all $t \in [t_0 - a, t_0 + a]$ such that $(y_n(t), t) \in \bar{B}_b(y_0) \times [t_0 - h, t_0 + h]$. Note that S_n is not empty. We claim that $[t_0 - h, t_0 + h] \subset S_n$. □