1 Recap

1. Uniqueness.
2. Statement of Picard's theorem (first version).

2 General existence and uniqueness theory

Theorem 1. *Let $D \subset \mathbb{R}^{n+1}$ be a domain and let $f : D \to \mathbb{R}^n$ be continuous (jointly) on D and Lipschitz in y on D with Lipschitz constant α. Let $R = \bar{B}_b(y_0) \times [t_0 - a, t_0 + a] \subset D$. Let $M = \max_R |f|$ and $h = \min\left(a, \frac{b}{M}, \frac{1}{2\alpha}\right)$. In the initial-value-problem (IVP) $y' = f(y,t)$ with $y(t_0) = y_0$ has a unique solution on $[t_0 - h, t_0 + h]$.*

Proof.

$$y_{n+1}(t) = y_0 + \int_{t_0}^{t} f(y_n(s), s)ds \tag{1}$$

with $y_0(t) = y_0$. Suppose S_n is the set of all $t \in [t_0 - a, t_0 + a]$ such that $(y_n(t), t) \in \bar{B}_b(y_0) \times [t_0 - h, t_0 + h]$. Note that S_n is not empty. We claim that $[t_0 - h, t_0 + h] \subset S_n$. We shall prove this claim by induction on n. Clearly it is true for $n = 0$. Suppose it is true until $n - 1$. Then $y_n(t) - y(t_0) = \int_{t_0}^{t} f(y_{n-1}(s), s)ds$. Thus $\|y_n(t) - y_0\| \leq Mh \leq b$ if $t \in [t_0 - h, t_0 + h] \subset S_{n-1}$. Thus it is true for n.
As mentioned earlier,

$$\|y_{n+1}(t) - y_n(t)\| \leq \alpha \left| \int_{t_0}^{t} \|y_n - y_{n-1}\| ds \right|, \tag{2}$$

provided $(y_n(s), s), (y_{n+1}(s), s) \in D$ for all s lying between t_0 and t.
As a consequence on $[t_0 - h, t_0 + h]$,

$$\begin{aligned} \|y_{n+1} - y_n\| &\leq \alpha \int_{t_0}^{t} \|y_n - y_{n-1}\| ds \\ \Rightarrow \|y_{n+1} - y_n\|_{C^0[t_0-h,t_0+h]} &\leq \alpha h \|y_n - y_{n-1}\|_{C^0[t_0-h,t_0+h]} \\ \leq \frac{1}{2}\|y_n - y_{n-1}\|_{C^0[t_0-h,t_0+h]} &\leq \frac{1}{2^n}\|y_1 - y_0\|_{C^0[t_0-h,t_0+h]}. \end{aligned} \tag{3}$$

Since $y_n = y_0 + (y_1 - y_0) + (y_2 - y_1)\ldots$, we see that by the Weierstrass M-test, this series converges uniformly to some limit y which is a continuous function. Since f is continuous, $\int_{t_0}^{t} f(y_n(s), s)ds$ converges to $\int_{t_0}^{t} f(y(s), s)ds$ by uniform convergence (why?). Thus the limiting y satisfies $y = y_0 + \int_{t_0}^{t} f(y(s), s)ds$. By the fundamental theorem of calculus, y is differentiable and satisfies the IVP. □

Theorem 2. *In the above theorem, h can be chosen to be* $\min\left(a, \frac{b}{M}\right)$.

Proof. Indeed, we claim that $\|y_n - y_{n-1}\| \leq \frac{M}{\alpha}\frac{|(\alpha(t-t_0)))^n|}{n!}$ for all $n \geq 1$. Indeed, for $n = 1$, it is easy. Assume inductively that it is true for $1, 2, \ldots, n$. Then

$$\begin{aligned} \|y_{n+1} - y_n\| &\leq \alpha \int_{t_0}^{t} \|y_n - y_{n-1}\| ds \\ &\leq \alpha \int_{t_0}^{t} \frac{M}{\alpha} \frac{|(\alpha(s-t_0)))^n|}{n!} ds \\ &= \frac{M}{\alpha} \frac{|(\alpha(t-t_0)))^{n+1}|}{(n+1)!}. \end{aligned} \tag{4}$$

Since the series $\sum_n \frac{M}{\alpha}\frac{(\alpha h))^n}{n!}$ converges (why?), by the Weierstrass M-test, we are done as above (why?) □

We now wish to characterise the maximal interval of existence. Here is a version of such a theorem (we are not stating it in the greatest generality possible but the technique of proof is generally applicable).

Theorem 3. *Let $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be locally Lipschitz. There exists a unique differentiable solution $y : (h_1, h_2) \to \mathbb{R}^n$ to $y' = f(y(t), t)$ with $y(t_0) = y_0$ where $h_1 < t_0 < h_2$ are extended real numbers such that (h_1, h_2) is the maximal interval of existence (what does this mean?). Moreover, if h_2 is finite, there exists a sequence $t_n \to h_2$ (with $t_n \in (h_1, h_2)$) such that $\|y(t_n)\| \to \infty$ and likewise for h_1.*

In other words, as long as y stays bounded, we can "continue" further. Equivalently, being unbounded is the only thing that can go wrong (this is in stark contrast to partial differential equations where higher order problems can play a role).

Proof. There surely is a solution on $[t_0 - h, t_0 + h]$ for some $h > 0$. Define h_2 as the supremum of all a_2 such that there is a solution on $[t_0 - h, a_2]$. By uniqueness there is a unique solution on $[t_0 - h, h_2)$. Likewise we can define h_1 and come up with the maximal interval of existence. To be continued.... □