1 Recap

1. Picard's theorem (second version).
2. Maximal interval of existence (stated the theorem and proved it exists).

2 General existence and uniqueness theory

Theorem 1. *Let $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be locally Lipschitz. There exists a unique differentiable solution $y : (h_1, h_2) \to \mathbb{R}^n$ to $y' = f(y(t), t)$ with $y(t_0) = y_0$ where $h_1 < t_0 < h_2$ are extended real numbers such that (h_1, h_2) is the maximal interval of existence (what does this mean?). Moreover, if h_2 is finite, there exists a sequence $t_n \to h_2$ (with $t_n \in (h_1, h_2)$) such that $\|y(t_n)\| \to \infty$ and likewise for h_1.*

Proof. Cont'd..... Suppose h_2 is finite and $\|y(t)\| \leq C$ on $[t_0, h_2)$. (Why is this the negation of the hypothesis in the theorem?) Then since $y' = f(y(t), t)$, we see that $\|y'\| \leq C$ on $[t_0, h_2)$- why? (note that the constant C can vary from inequality to inequality) As a consequence, $\|y(s) - y(t)\| \leq |t - s|C$ and hence $y(h_2) := \lim_{t \to h_2^-} y(t)$ exists (why?). Because of uniform convergence, from $y(t) = y_0 + \int_{t_0}^t f(y, s)ds$ we see that $y(h_2) = y_0 + \int_{t_0}^{h_2} f(y, s)ds$. By the fundamental theorem of calculus, y is differentiable at h_2 and satisfies the ODE there. Thus we can extend the solution further using the existence theorem and arrive at a contradiction for maximality. □

We can apply this result to prove that non-autonomous linear systems $y' = A(t)y + B(t)$ with $y(t_0) = y_0$ have unique solutions on $(-\infty, \infty)$ if A, b are C^1 on $\mathbb{R}$. Indeed, $f(y, t) = A(t)y + B(t)$. Now f is C^1 and hence locally Lipschitz. Thus if there is a solution, it is unique. Now by the existence theorem, the solution exists on some maximal interval (h_1, h_2). Note that if either of them (WLog h_2) is finite, then since $\|y(t)\| \leq C + \int_{t_0}^t (\|A(s)\| \|y(s)\| + \|B(s)\|)ds \leq C(1 + \int_{t_0}^t \|y(s)\|)$ on $[t_0, h_2)$, by Gronwall, y is bounded and hence by the previous theorem, we have a contradiction.
We can actually prove a more general existence theorem due to Peano.

Theorem 2. *Let f be continuous on a rectangle $R = \bar{B}_b(y_0) \times [t_0 - a, t_0 + a]$. Then there exists a solution (possibly non-unique) to $y' = f(y, t)$ with $y(t_0) = y_0$ on $[t_0 - h, t_0 + h]$ where $h = \min(a, b/M)$ where $M = \max_R |f|$.*

There are two proofs using approximations.

1. Using actual solutions to an approximation of the problem: Using the Stone-Weierstrass theorem (Let X be compact and Hausdorff. Let A be a subalgebra of $C(X, \mathbb{R})$ which contains a non-zero constant function. Then A is dense in $C(X, \mathbb{R})$ iff it separates points), we see that there is a sequence of smooth functions $f_n \to f$ on R uniformly.
2. Using approximate solutions to the original problem

Theorem 3. *Let f be continuous on a rectangle $R = \bar{B}_b(y_0) \times [t_0 - a, t_0 + a]$. Then there exists a solution (possibly non-unique) to $y' = f(y, t)$ with $y(t_0) = y_0$ on $[t_0 - h, t_0 + h]$ where $h = \min(a, b/M)$ where $M = \max_R |f|$.*

There are two proofs using approximations.

1. Using actual solutions to an approximation of the problem: Using the Stone-Weierstrass theorem (Let X be compact and Hausdorff. Let A be a subalgebra of $C(X, \mathbb{R})$ which contains a non-zero constant function. Then A is dense in $C(X, \mathbb{R})$ iff it separates points), we see that there is a sequence of smooth functions $f_n \to f$ on R uniformly. (Another possibility is convolution with a nice function but that approximates only on a slightly smaller domain.) Now solve $y_n' = f_n(y_n, t)$ with $y_n(t_0) = y_0$ and $(y_n(t), t) \in \bar{B}_b(y_0) \times [t_0 - h_n, t_0 + h_n]$ where $h_n = \min(a, b/M_n)$ (this property follows from the proof). Note that $\|y_n'\| \leq M_n$. Thus $\|y_n(t) - y_n(s)\| \leq M_n \|t - s\|$ for all $t, s \in [t_0 - h_n, t_0 + h_n]$. Given $\epsilon > 0$, we can choose n large enough so that $|h_n - h| < \epsilon$. Thus for all such n, y_n is uniformly equicontinuous and uniformly bounded on $I_\epsilon = [t_0 - h + \epsilon, t_0 + h - \epsilon]$. To be continued.....