1 Recap

1. Maximal interval.
2. First proof of Peano - construction of y_n.

2 General existence and uniqueness theory

Theorem 1. *Let f be continuous on a rectangle $R = \bar{B}_b(y_0) \times [t_0 - a, t_0 + a]$. Then there exists a solution (possibly non-unique) to $y' = f(y,t)$ with $y(t_0) = y_0$ on $[t_0 - h, t_0 + h]$ where $h = \min(a, b/M)$ where $M = \max_R |f|$.*

Proof. There are two proofs but the first one proves something slightly weaker, i.e., existence on $(t_0 - h, t_0 + h)$.

1. Made a mistake. Will do the next time.
2. Using approximate solutions to the original problem: The idea is to use the Euler method to produce approximate solutions and hope that they converge (using Arzela-Ascoli again). Here is the precise definition of an ϵ-approximate solution: Let f be defined and continuous on a domain $D \subset \mathbb{R}^{n+1}$. An ϵ-approximate solution on $I = [t_0 - a, t_0 + a]$ is a function $y : I \to \mathbb{R}^n$ such that

   (a) $(t, y(t)) \in D$ for all $t \in I$.

   (b) y is C^1 on I except possibly for a finite set $S \subset I$ (but it has left and right derivatives on S).

   (c) $\|y' - f(y,t)\| \leq \epsilon$ on $I \cap S^c$.

   We produce ϵ-approximate solutions on $[t_0 - h, t_0 + h]$ for $h = \min(a, b/M)$ where f is continuous on $R = \bar{B}_b(y_0) \times [t_0 - a, t_0 + a]$ and $M = \max_R f$: We shall construct it on $[t_0, t_0 + h]$ (and similar construction works on the other side). Divide the interval into subintervals and on each subinterval $[t_k, t_{k+1}]$ we give a linear approximation using the Euler method, i.e., Solve $z' = f(z_{k-1}, t_{k-1}), z(t_{k-1}) = z_{k-1}$ to get $z_k = z(t_k) = z_{k-1} + f(z_{k-1}, t_{k-1})(t_k - t_{k-1})$. But we need to choose the subintervals carefully so that these piecewise linear solutions all lie in R. Firstly, since f is uniformly continuous on R, $\|f(y,t) - f(\tilde{y}, \tilde{t})\| < \epsilon$ whenever $\|t - \tilde{t}\| + \|y - \tilde{y}\| \leq \delta < b$. Define $y(t)$ as the piecewise linear function given by $z(t)$ on each interval. We shall divide the interval into equal parts of size at most δ_1 which we shall choose later (it will turn out that $\delta_1 = \min(\delta, \delta/M)$ works).
   First we prove by induction on i that $(y(t), t) \in R$ on $[t_{i-1}, t_i]$. For $i = 1$, $\|y(t) - y(t_0)\| = |t - t_0| \|f(y_0, t_0)\| \leq M\delta_1 < \delta < b$ and hence it is true for $i = 1$. Assume truth for $1, 2, \ldots, i$. Then for $i+1$, $\|y(t) - y(t_0)\| \leq \|y(t) - y(t_i)\| + \|y(t_i) - y(t_{i-1})\| + \ldots \leq M\|t - t_0\| \leq Mh \leq b$. Thus we are done.
   Using the same method one can show that $(y(t), t)$ is an ϵ-approximate solution. We have produced approximate solutions. Now choose $\epsilon_n = \frac{1}{n}$. The corresponding approximate solutions y_n are uniformly bounded and uniformly equicontinuous (why?) and hence by Arzela-Ascoli, a subsequence (that we shall abuse

notation and continue to denote as) y_n converges uniformly to a continuous function y. Uniform convergence now implies that y is a solution to the IVP. (Why?)

□

The next order of business to see how the solution depends on initial data as well as on other parameters that may occur in the differential equation (like physical constants for instance).

Theorem 2. *Let R be a "rectangle" as in Peano's theorem. Let $f, \tilde{f}$ be continuous on R, and uniformly Lipschitz in y with constants $\alpha, \tilde{\alpha}$. Let $y, \tilde{y}$ be the solutions of $y' = f(y,t), y(t_0) = y_0$, and $\tilde{y}' = \tilde{f}(\tilde{y}, t), \tilde{y}(\tilde{t}_0) = \tilde{y}_0$ in some closed interval I (containing $t_0, \tilde{t}_0$ and contained in R) of length $|I|$. Then*

$$\max_I \|y(t) - \tilde{y}(t)\| \leq e^{\min(\alpha,\tilde{\alpha})|I|} \left(\|y_0 - \tilde{y}_0\| + |I| \max_R \|f - \tilde{f}\| + \max(\|f\|_{\max_R}, \|\tilde{f}\|_{\max_R}) |t_0 - \tilde{t}_0| \right) \tag{1}$$

Proof.

$$\begin{aligned}
y - \tilde{y} &= y(t_0) - \tilde{y}(t_0) + \int_{t_0}^{t} (f(y(s),s) - \tilde{f}(\tilde{y}(s),s))ds \\
&= y(t_0) - \tilde{y}(\tilde{t}_0) + \tilde{y}(\tilde{t}_0) - \tilde{y}(t_0) + \int_{t_0}^{t} (f(y(s),s) - \tilde{f}(\tilde{y}(s),s))ds \\
\Rightarrow \|y - \tilde{y}\| &\leq \|y(t_0) - \tilde{y}(\tilde{t}_0)\| + \|\tilde{y}(\tilde{t}_0) - \tilde{y}(t_0)\| + \int_{t_0}^{t} \|f(y(s),s) - \tilde{f}(\tilde{y}(s),s)\| ds \\
&\leq \|y_0 - \tilde{y}_0\| + \max_I \|\tilde{y}'\| |t_0 - \tilde{t}_0| + \int_{t_0}^{t} \|f(y(s),s) - \tilde{f}(\tilde{y}(s),s)\| ds \\
&\leq \|y_0 - \tilde{y}_0\| + \max_R \|\tilde{f}\| |t_0 - \tilde{t}_0| + \int_{t_0}^{t} \|f(y(s),s) - \tilde{f}(y(s),s)\| ds + \int_{t_0}^{t} \|\tilde{f}(y(s),s) - \tilde{f}(\tilde{y}(s),s)\| ds \\
&\leq \|y_0 - \tilde{y}_0\| + \max_R \|\tilde{f}\| |t_0 - \tilde{t}_0| + |I| \max_R \|f - \tilde{f}\| + \tilde{\alpha} \int_{t_0}^{t} \|y - \tilde{y}\| ds. \qquad (2)
\end{aligned}$$

Thus by Gronwall

$$\|y - \tilde{y}\| \leq e^{\tilde{\alpha}|I|} \left(\|y_0 - \tilde{y}_0\| + \max_R \|\tilde{f}\| |t_0 - \tilde{t}_0| + |I| \max_R \|f - \tilde{f}\| \right) \tag{3}$$

Interchanging the roles of $y, \tilde{y}$, we are done. □

As a consequence, the solution depends continuously on the initial data and parameters involved. We can prove more. In fact, we can prove that if f is smooth, so is y. The rough idea is (for proving differentiability) to first pretend differentiability holds, deduce the ODE for the derivative, write an ODE for the difference quotient, subtract these two ODE and use the Gronwall inequality to deduce that indeed the difference quotient converges to the (hypothetical) derivative.