1 Recap

1. Second proof of Peano. First proof needed to be corrected: We had produced y_n on $[t_0 - h_n, t_0 + h_n]$ and noted that $h_n \to h$. Now consider the intervals $I_n = [t_0 - h + \frac{1}{n}, t_0 + h + \frac{1}{n}]$. Fix I_m (for large enough m) and consider the sequence y_n. It is bounded and uniformly equicontinuous. Hence, by Arzela-Ascoli, there exists a uniformly convergent subsequence y_{k1} (converging to Y_1). Now on I_{m+1}, there exists a subsequence y_{k2} of y_{k1} that converges to Y_2. Repeat this process. Note that Y_2 agrees with Y_1 on I_m and so on. Thus we have a limiting continuous function y on $I_\infty = (t_0 - h, t_0 + h)$ which satisfies the IVP (by FTC). This result is slightly weaker than the one proven by the second method.

2. Stated continuous dependence on parameters.

2 General existence and uniqueness theory

Theorem 1. *Let R be a "rectangle" as in Peano's theorem. Let $f, \tilde{f}$ be continuous on R, and uniformly Lipschitz in y with constants $\alpha, \tilde{\alpha}$. Let $y, \tilde{y}$ be the solutions of $y' = f(y,t), y(t_0) = y_0$, and $\tilde{y}' = \tilde{f}(\tilde{y}, t), \tilde{y}(\tilde{t}_0) = \tilde{y}_0$ in some closed interval I (containing $t_0, \tilde{t}_0$ and contained in R) of length $|I|$. Then*

$$\max_I \|y(t) - \tilde{y}(t)\| \leq e^{\min(\alpha,\tilde{\alpha})|I|} \left(\|y_0 - \tilde{y}_0\| + |I| \max_R \|f - \tilde{f}\| + \max(\|f\|_{\max_R}, \|\tilde{f}\|_{\max_R}) |t_0 - \tilde{t}_0| \right) \tag{1}$$

Proof.

$$\begin{aligned}
y - \tilde{y} &= y(t_0) - \tilde{y}(t_0) + \int_{t_0}^{t} (f(y(s),s) - \tilde{f}(\tilde{y}(s),s))ds \\
&= y(t_0) - \tilde{y}(\tilde{t}_0) + \tilde{y}(\tilde{t}_0) - \tilde{y}(t_0) + \int_{t_0}^{t} (f(y(s),s) - \tilde{f}(\tilde{y}(s),s))ds \\
\Rightarrow \|y - \tilde{y}\| &\leq \|y(t_0) - \tilde{y}(\tilde{t}_0)\| + \|\tilde{y}(\tilde{t}_0) - \tilde{y}(t_0)\| + \int_{t_0}^{t} \|f(y(s),s) - \tilde{f}(\tilde{y}(s),s)\|ds \\
&\leq \|y_0 - \tilde{y}_0\| + \max_I \|\tilde{y}'\| |t_0 - \tilde{t}_0| + \int_{t_0}^{t} \|f(y(s),s) - \tilde{f}(\tilde{y}(s),s)\|ds \\
&\leq \|y_0 - \tilde{y}_0\| + \max_R \|\tilde{f}\| |t_0 - \tilde{t}_0| + \int_{t_0}^{t} \|f(y(s),s) - \tilde{f}(y(s),s)\|ds + \int_{t_0}^{t} \|\tilde{f}(y(s),s) - \tilde{f}(\tilde{y}(s),s)\|ds \\
&\leq \|y_0 - \tilde{y}_0\| + \max_R \|\tilde{f}\| |t_0 - \tilde{t}_0| + |I| \max_R \|f - \tilde{f}\| + \tilde{\alpha} \int_{t_0}^{t} \|y - \tilde{y}\|ds. \qquad (2)
\end{aligned}$$

Thus by Gronwall

$$\|y - \tilde{y}\| \leq e^{\tilde{\alpha}|I|} \left(\|y_0 - \tilde{y}_0\| + \max_R \|\tilde{f}\| |t_0 - \tilde{t}_0| + |I| \max_R \|f - \tilde{f}\| \right) \tag{3}$$

Interchanging the roles of $y, \tilde{y}$, we are done. □

As a consequence, the solution depends continuously on the initial data and parameters involved. We can prove more. In fact, we can prove that if f is smooth, so is y. The rough idea is (for proving differentiability) to first pretend differentiability holds, deduce the ODE for the derivative, write an ODE for the difference quotient, subtract these two ODE and use the Gronwall inequality to deduce that indeed the difference quotient converges to the (hypothetical) derivative.

We shall now prove that if f and $y_0 = y(0)$ are C^1 functions of $v \in \mathbb{R}^m$ then so is y. All we need to do is to prove that $\frac{\partial y}{\partial v_i}$ exists and is continuous for all i. Fix i and call v_i as w. We need to identify the (hypothetical) derivative y_w and prove that indeed it is the correct derivative. To this end, consider the following ODE obtained by differentiating the original IVP. (Later we will show that $u = y_w$.) Let's work with one function y (as opposed to a vector) for simplicity. The proof doesn't change much otherwise.

$$\begin{aligned} u' &= f_y u + f_w \\ u(0) &= (y_0)_w. \end{aligned} \tag{4}$$

Now this ODE is a linear system for u. So it has a unique differentiable (in t) solution for as long as the right-hand-side makes sense. Moreover, the right-hand-side is continuous and hence u is continuous jointly in t, v.
Consider the equation satisfied by the difference-quotient $\Delta_h y = \frac{y(t,v_1,\ldots,w+h,\ldots)-y(t,v_1,\ldots,w,\ldots)}{h}$.

$$\begin{aligned} (\Delta_h y)' &= \frac{f(y(t,\ldots,w+h,\ldots),t,\ldots,w+h,\ldots) - f(y(t,\ldots,w,\ldots),t,\ldots,w,\ldots)}{h} \\ \Delta_h y(0) &= \Delta_h(y_0). \end{aligned} \tag{5}$$

Now subtract to get

$$(\Delta_h y - u)' = \frac{f(y(t,v) + h\Delta_h y, t, \ldots, w+h, \ldots) - f(y(t,v),t,v)}{h} - f_y u - f_w \tag{6}$$

$$\Delta_h y(0) - u(0) = \Delta_h(y_0) - (y_0)_w \tag{7}$$

Thus

$$\begin{aligned} &\|\Delta_h y - u\| \leq \|\Delta_h(y_0) - (y_0)_w\| \\ &+ \int_0^t \left\| \frac{f(y(s,v) + \Delta_h y, s, \ldots, w+h, \ldots) - f(y(s,v),s,v)}{h} - f_y u - f_w \right\| ds \\ &\leq \|\Delta_h(y_0) - (y_0)_w\| + \int_0^t \Big(\|f_y(\xi) - f_y\| \|\Delta_h y\| \\ &+ C\|\Delta_h y - u\| + \|f_w(\xi) - f_w\| \Big) ds \end{aligned} \tag{8}$$

To be continued....