1 Recap

1. Smooth and continuous dependence on parameters theorems.

2 Differentiable dependence on parameters

Continuing the proof from the last time,

$$\begin{aligned}\|\Delta_h y - u\| &\leq \|\Delta_h(y_0) - (y_0)_w\| + \int_0^t \Big(\|f_y(\xi) - f_y\|(\|\Delta_h y - u\| + \|u\|) \\ &\quad + C\|\Delta_h y - u\| + \|f_w(\xi) - f_w\|\Big) ds \\ &\leq \epsilon + C\int_0^t \|\Delta_h y - u\| ds, \end{aligned} \tag{1}$$

for all $0 < \|h\| < \delta$ (depending on ϵ). By Gronwall, we can see that indeed y is partially differentiable and u is its derivative w.r.t w (why?) Now since u is continuous, y is C^1. □

3 A little bit about numerical methods

It is crucial to find efficient algorithms to solve ODE on computers. (After all, we know most of these explicit methods are slow if nothing else.) We want to solve $y' = f(y,t)$ (where y, f are vector-valued) with $y(t_0) = y_0$. By "solve", we want $y(t_1)$ for any given t_1 quickly (Wlog $t_1 > t_0$) and approximately to a given error ϵ. We are "given" $f(y,t)$ in that we can assume that there is a subroutine that provides $f(y,t)$ quickly and exactly to us. (Actually, it cannot be given exactly but it is easiest to assume this hypothesis.) We assume that $f(y,t)$ is smooth (for simplicity) on $\mathbb{R}^{n+1}$. Thus there exists a unique solution for some time (p,q). We assume that t_1 lies in this interval. We also assume that on $[t_0,t_1]$, $\|y''\| \leq M$, and that f is actually Lipschitz on $\mathbb{R}^{n+2}$ in y with constant L.

The simplest algorithm is due to Euler: Divide $[t_0,t_1]$ into equal pieces of size h. Then define $y_n = y_{n-1} + f(y_{n-1},t_{n-1})h$. We hope that if h is sufficiently small, then $\|y_N - y(t_1)\| < \epsilon$ where y is the actual solution and $N = \frac{t_1-t_0}{h}$. Now $y(t_n) = y(t_{n-1}) + \int_{t_{n-1}}^{t_n} f(y(s),s)ds$. Now $\|y(t_n) - y(t_{n-1}) - hy'(t_{n-1})\| \leq Mh^2$ (Taylor's theorem). Thus $\|y(t_n) - y(t_{n-1}) - hf(y(t_{n-1}),t_{n-1})\| \leq Mh^2$. Now

$$\begin{aligned}\|y(t_n) - y_n\| &\leq Mh^2 + \|y_{n-1} + hf(y_{n-1},t_{n-1}) - y(t_{n-1}) - hf(y(t_{n-1},t_{n-1}))\| \\ &\leq \|y_{n-1} - y(t_{n-1})\|(1+Lh) + Mh^2 \\ \Rightarrow \|y(t_n) - y_n\| &\leq \frac{M((1+Lh)^n - 1)}{L} h \\ \Rightarrow \|y_N - y(t_1)\| &\leq \frac{Mh}{L}(e^{(t_1-t_0)L} - 1). \end{aligned} \tag{2}$$

The bottom line is that the error is linear in h. (Hence Euler is sometimes called a first-order method.) The problem with Euler's method is not just that it is slow. It is unstable. For instance, if $y' = -ky$, then y decays exponentially. However, depending on the step size h, the numerical solution can oscillate and do other crazy things. Do note that rounding-off errors play a big role too. For instance if h is too small, then while truncation errors are small, the machine rounding off errors can be large. (We add lots of small numbers.) So one needs to resort to complicated summation approaches too. (Like compensated summation.) The other thing is that f itself may not be specified exactly and that also introduces errors.

A better method (at least as far as truncation error is concerned) is the midpoint method: It uses the observation that $y'(t+h/2) \approx \frac{y(t+h)-y(t)}{h}$. Moreover, $y(t+h/2) \approx y(t) + \frac{h}{2}y'(t)$. Thus, $y_{n+1} = y_n + hf(t_n + h/2, y_n + \frac{h}{2}f(y_n, t_n))$. The global truncation error is roughly $O(h^2)$. Indeed, $y'(t+h/2) - \frac{y(t+h)-y(t)}{h} = O(h^2)$ and hence in each step the error is $O(h^3)$. So the global error is roughly $O(h^2)$. A rough idea behind why the midpoint method works: $\int_t^{t+h} y'(s)ds - y'(t+h/2)h = \int_t^{t+h}(y'(t) + y''(t)(s-t) + O((s-t)^2)ds - y'(t) - y''(t)h^2/2 + O(h^3) = O(h^3)$.
This leads to the (explicit) Runge-Kutta methods: $y_{n+1} = y_n + h\sum b_i k_i$ where $k_1 = f(t_n, y_n)$, $k_2 = f(t_n + c_2 h, y_n + ha_{21}k_1)$, $k_3 = f(t_n + c_3 h, y_n + h(a_{31}k_1 + a_{32}k_2))$, and so on. (There are implicit methods as well.) A popular choice is the RK4 method. Basically, we can try a variant of the midpoint method, i.e., $c_2 = c_3 = \frac{1}{2}, c_1 = 0, c_4 = 1$. The simplest thing is to assume $a_{21} = a_{31} = \frac{1}{2}$, $a_{41} = 1$ and the rest as 0. Then we can determine the b_i by Taylor series (upto order 4) and it turns out $b_1 = \frac{1}{6} = b_4$, $\frac{b_2}{=}\frac{2}{6} = b_3$.

4 Sturm-Liouville theory

Consider a vibrating elastic rod of density $\rho(x)$ and tension $k(x)$. Then suppose we take an infinitesimal element dx between $x, x+dx$. Suppose it moves up by $y(x)$. Then the net vertical force on it is $k(x+dx)\sin(\theta(x+dx)) - k(x)\sin(\theta(x)) = dk\sin(\theta(x)) + k(x)\cos(\theta(x))dx = \rho(x)dx\frac{\partial^2 y}{\partial t^2}$. Thus

$$\frac{\partial}{\partial x}\left(k(x)\sin(\theta(x))\right) = \rho(x)\frac{\partial^2 y}{\partial t^2} \tag{3}$$

For small θ, $\sin(\theta) \approx \frac{\partial y}{\partial x}$. Thus we get the wave equation

$$\frac{\partial}{\partial x}\left(k(x)\frac{\partial y}{\partial x}\right) = \rho(x)\frac{\partial^2 y}{\partial t^2}. \tag{4}$$

Now we substitute $y = u(x)\cos(\nu t)$ to get

$$(ku')' = -\rho\nu^2 u(x). \tag{5}$$

For a finite rod $x \in [a, b]$ here are some natural boundary conditions:
$u(a) = u(b) = 0$ (rigid ends), $u'(a) + \alpha u(a) = u'(b) + \beta u(b) = 0$ (elastically held ends), $u(a) = u(b), u'(a) = u'(b)$ (periodic boundary conditions).
For example if k, ρ are constants (set to 1 by choosing units appropriately), and $a = 0, b = \pi$, then $u'' = -\nu^2 u$ with $u(0) = u(\pi) = 0$. Thus $u = A\sin(\nu x)$ where ν is an integer.