1 Recap

1. Midpoint method.
2. Elastic rod. Example $u'' = -\nu^2 u$ with rigid ends.

2 Sturm-Liouville theory

Thus the "eigenvalues" of the corresponding linear map from the vector space of smooth functions to itself are discrete and have no limit points. Moreover, it turns out that any "reasonable function" can be "written using sines and cosines". This theorem of Fourier series needs Lebesgue integration and a little bit of functional analysis to state and prove.
Motivated by this example we study the general *regular/elliptic* Sturm-Liouville problem:

$$Lu = -(pu')' + qu = \lambda \rho u, \tag{1}$$

where $p > 0, \rho > 0$, p is in C^1, q, ρ are continuous and $\lambda \in \mathbb{R}$ on $[a, b]$. The boundary conditions will be of the form

$$\begin{aligned} \alpha_1 u(a) + \alpha_2 u'(a) = \alpha_3,\ |\alpha_1| + |\alpha_2| > 0 \\ \beta_1 u(b) + \beta_2 u'(b) = \beta_3,\ |\beta_1| + |\beta_2| > 0. \end{aligned} \tag{2}$$

We now have the following theorem (akin to its finite-dimensional counterpart).

Theorem 1. *Consider the BVPs:*

$$Lu = r, \tag{3}$$

where r is continuous on $[a, b]$, and

$$Lu = 0,\ \alpha_3 = 0 = \beta_3. \tag{4}$$

Then the following hold (Fredholm's alternatives).

1. *If 4 has only a trivial solution, then 3 has a unique solution.*
2. *If 4 has a non-trivial solution, then 3 has infinitely many solutions, if it has one solution.*

Proof. 1. If u_1, u_2 are two solutions of 3, then $u_1 - u_2$ is a solution of 4. Hence if there is a solution, it is unique. Now we prove existence. We can prove that there is a unique solution to $Lu = r$ with $u(a) = c_1, u'(a) = c_2$ (Exercise). Given this fact, we have two linearly independent solutions u_1, u_2 with $u_1(a) = 1 = u_2'(a)$ and $u_1'(a) = u_2(a) = 0$. Thus $u = c_1 u_1 + c_2 u_2$ in general. We simply need to choose c_1, c_2 so that the boundary conditions are met. This will happen if the matrix $A = \begin{bmatrix} \alpha_1 u_1(a) + \alpha_2 u_1'(a) & \alpha_1 u_2(a) + \alpha_2 u_2'(a) \\ \alpha_1 u_1(b) + \alpha_2 u_1'(b) & \alpha_1 u_2(b) + \alpha_2 u_2'(b) \end{bmatrix}$ is invertible. If A is not invertible, then its kernel will produce c_1, c_2 such that $c_1 u_1 + c_2 u_2$ is a non-trivial solution of 4.

2. Indeed, if u_0 is a solution and u is a non-trivial solution of the homogeneous problem, then $u_0 + cu$ is a solution for all $c \in \mathbb{R}$.

□

We now prove some properties about the linear map L.

Theorem 2. *1. L is symmetric, i.e., $\int_a^b Luv = \int_a^b uLv$ (for u, v satisfying the boundary conditions with $\alpha_3 = \beta_3 = 0$).*

2. Suppose u, v are C^2 functions that satisfy the "eigenvalue" equations $Lu = \lambda\rho u$, $Lv = \mu\rho v$ where $\lambda \neq \mu$ (and the boundary conditions), then u, v are "orthogonal" in the sense that $\int_a^b \rho uv = 0$.

Proof. The key point is to prove Lagrange's identity: For any two C^2 functions u, v,

$$vLu - uLv = (pW)', \tag{5}$$

where $W = uv' - u'v$ (why?) Thus upon integration, we see that L is symmetric because $W(a) = W(b) = 0$ from the boundary conditions. Now $0 = \lambda \int_a^b \rho uv - \mu \int_a^b \rho uv$ and hence u, v are orthogonal. □