1 Recap

1. Fredholm's alternative.
2. Symmetry of the SL operator.

2 Sturm-Liouville theory

One of our main aims is to prove the existence of eigenfunctions.

Theorem 1. *The eigenvalues of the regular Sturm-Liouville problem (with $\alpha_3 = \beta_3 = 0$) is an infinite sequence $\lambda_0 < \lambda_1 \ldots$ converging to ∞. The eigenfunction u_n corresponding to λ_n is unique upto a constant factor and has exactly n zeroes in (a, b).*

We can of course find a solution on $[a, b]$ satisfying the boundary condition at a. The challenge is to make sure that the boundary condition at b is also met. The idea will be to take all possible initial conditions at a (satisfying the boundary condition at a), and seeing how this 1-parameter family of u behaves at b. We shall show that the boundary values vary between all possible values (via the intermediate value theorem). We also need to study the "oscillation" of u to know how many zeroes exist. Uniqueness is actually straightforward. Suppose u, v are eigenvectors with the same eigenvalue. Then the boundary condition at a implies that the Wronskian at a is 0 and hence is zero throughout. Thus they are linearly dependent (from a previous HW problem).

Prior to studying the oscillations and boundary values, we prove the following basic result: Any non-trivial solution u can have at most finitely many zeroes in $[a, b]$. Indeed, suppose not. Consider $\xi_n \to \xi \in [a, b]$. By continuity, $u(\xi) = 0$. Now $0 = \frac{u(\xi+\xi_n-\xi)-u(\xi)}{\xi_n-\xi}$. Thus $u'(\xi) = 0$ and by uniqueness, u is identically 0.

We now introduce an important tool to study oscillations and boundary values. Note that the boundary conditions are such that it is enough to prescribe $\frac{u(b)}{u'(b)}$ on the boundary (or perhaps $u'(b) = 0$). Motivated by this observation, we introduce the Prüfer substitution: $r = \sqrt{u^2 + p^2 u'^2}$, $\cos(\theta) = \frac{Pu'}{r}, \sin(\theta) = \frac{u}{r}$. Since either u or u' is non-zero for a non-trivial solution, r is well-defined and is C^1. On the other hand, $\theta(t) : [a, b] \to \mathbb{R}$ is more delicate. If you know covering maps, then you simply choose an initial θ_0 and consider the lift of the map to the universal cover. It is a unique continuous map which is actually C^1 because of the way the lift is constructed. If you don't know covering maps, the basic idea is as follows. Firstly, choose some θ_0 for $t = a$. Now one can surely locally uniquely find $\theta(t)$ for some short period of time (why?) which is continuous (in fact C^1). Now cover the compact interval $[a, b]$ with finitely many such open sets to determine $\theta(t)$. Uniqueness follows (why?)

By differentiation (how?), we can prove that

$$\theta' = (\lambda\rho - q)\sin^2\theta + \frac{1}{p}\cos^2(\theta) = F(t, \theta),$$
$$r' = \frac{1}{2}\left(\frac{1}{p} - (\lambda\rho - q)\right) r\sin(2\theta). \tag{1}$$

Once θ is known,

$$r = r(a) \exp\left(\frac{1}{2}\int_a^t \left(\frac{1}{p(s)} - q(s)\right) \sin(2\theta(s)) ds\right).$$

The boundary conditions for the SL problem only specify boundary conditions for θ. Not for r. So given a solution of the SL BVP, we get a family of solutions of the new system (where $r(a)$ is arbitrary). Given a solution of the new system (with the right boundary conditions for θ), we get a solution of the SL BVP (why?). Note that changing $r(a)$ only scales u by a constant factor. Hence the zeroes of u can be located by studying θ.

Note that F is Lipschitz uniformly in θ and hence we obtain a unique solution for θ given the initial $\theta(a) = \gamma$ for a short period of time.