1 Logistics

Email: vamsipingali@iisc.ac.in. Course webpage: `http://math.iisc.ac.in/~vamsipingali/ma241ODE2025spring/ma241.html`. Quiz : 20% (Roughly once a week. Copying (from each other or the internet) is strictly not allowed. The best $n-2$ quizzes out of n quizzes will be considered), Midterm - 30%, and Final - 50%.

2 What is this course about and why should you care?

An ordinary differential equation (ODE) is an equation for a vector-valued function $\vec{F}(t, \vec{y}, \vec{y}', \vec{y}'', \ldots, \vec{y}^{(k)}) = 0$ for some function $\vec{F}$. (Typically, some initial/boundary conditions are specified.) In general, PDE (partial differential equations) and ODE together model all of reality. This course studies ODE, i.e., "explicit" ways of solving, existence and uniqueness theory, and dependence on parameters (stability analysis). In a way, the study of differential equations is akin to theology, i.e., the fundamental questions are those of existence and uniqueness. While the ideal situation is to be able to write a nice formula that can be implemented quickly on a computer, at the least, we want to know whether a solution exists and is unique in the first place. Here are examples/counterexamples:

1. $y' = 0$ on (a, b). Either using the mean-value-theorem or the fundamental theorem of calculus, $y = y_0$. Now if the domain is two disconnected intervals, what is the general solution?

2. $y' = y$ on (a, b). We can guess (using separation) that $y = Ae^x$ is a solution. Here is a proof that it is *the* general solution: $(ye^{-x})' = 0$ and hence $y = Ae^x$.

3. $y' = y^2$. Here the solution is $y = \frac{y_0}{ty_0 - 1}$. Note that the solution "blows up" at $t = C$ (as opposed to the previous examples).

4. $y' = 2\sqrt{y}$ with $y(0) = 0$. Note that $y = 0$ *and* $y = t^2$ satisfy this initial-value-problem!

5. $(y')^2 = -1$ obviously has no real solution.

As mentioned earlier, many things in real life are modelled by ODE. In fact, ODE theory is useful in other areas of pure maths too (like differential geometry). Here are examples:

1. Population growth: The most obvious model of population growth/radioactivity is $y' = ky$ whose unique solution is $y = Ae^{kx}$. Obviously population cannot grow indefinitely. A more realistic model is Logistic model $y' = ky(M - y)$ where M is called the carrying capacity (after the population reaches M, it decreases). Note that $y = 0, y = M$ are solutions (they are called fixed points). While one can separate and find a solution $y = \frac{y_0 M e^{Mkt}}{M - y_0 + y_0 e^{Mkt}}$, it is not obvious that it is unique. The same arguments as before show uniqueness. Note that if you start with $y_0 > 0$, eventually the population will converge to M. Thus M is a stable equilibrium and 0 is unstable.

2. Damped forced oscillator: $y'' = -ky - by' + f$ where $f(t)$ is the force function. Suppose $f = 0$, then we get $y'' = -ky - by'$. Solving this equation is not immediately obvious and we shall do it later (if you accept complex-valued functions (HW), then you can try $y = e^{At}$ where A is a complex number and get $A^2 + bA + k = 0$, i.e., $A = \frac{-b \pm \sqrt{b^2 - 4k}}{2}$. Usually, b is small compared to k and hence A is a genuine complex number. This leads to damped oscillations. Again, one must prove that indeed all solutions can be gotten using this procedure.

3. Moon lander: Let $h(t)$ be the height, $v = h'$, $m(t)$ the mass, $f(t)$ the thrust (obtained by firing tiny rockets). Assume that $0 \leq f(t) \leq 1$ (in appropriate units). Then $mh'' = -gm + f$ and $m' = kf$. This is sometimes called a system of ODE (as opposed to a single equation for a single function). Again we need to prove that solutions do exist, and are unique to begin with. Assuming this (and in fact, assuming we actually solve this equation), we have a further goal: The aim is to choose f so that if τ is the smallest time for which $h(\tau) = v(\tau) = 0$ (such a τ better exist!) (where $h \geq 0, m \geq 0$), then the fuel $m(\tau)$ is maximum. This problem is called a problem of optimal control. Control theory uses fancy mathematics like differential geometry to govern the motion of everyday things like drones and rockets.

4. Covid infection: Let S be susceptible individuals, E be exposed ones, I be infected ones, and R removed ones. Ignore births and non-covid deaths. Then $S + E + I + R = N$ (a constant large number). Let $\beta(t)$ be the contact rate, σ the infection rate, and γ the removal rate. Then the SEIR model for Covid is

$$\begin{aligned} S' &= -\beta \frac{SI}{N} \\ E' &= \beta \frac{SI}{N} - \sigma E \\ I' &= \sigma E - \gamma I \\ R' &= \gamma I. \end{aligned} \tag{1}$$

   If β is a constant, $R_0 = \frac{\beta}{\gamma}$ is the called (the infamous) reproduction number (the number of individuals an infected individual transmits to). This system of ODE also needs existence and uniqueness theory. After that, one needs to develop a fast algorithm to solve it (thankfully several such methods exist, one of them being the Runge-Kutta method).

5. Vector fields: Now for a pure maths example. Consider a smooth function $X : \mathbb{R}^2 \to \mathbb{R}^2$. Such a beast is called a smooth vector field. Then the flow of the vector field is $x_i' = X_i(x_1(t), x_2(t))$. Again this is a system of ODE. Bear in mind that many of the equations above fall in this framework. This flow is supposed to be like the trajectory of a paper boat placed in a river. Note that the flow need not exist for all time (like in the blowup example earlier). (Why is this a pure maths example? You will need to take a course on differential geometry to understand.) Vector fields also play a role in flow-based diffusion models to generate images.