

# 1 Recap

1. Proved that any solution to SL has only finitely many zeroes.
2. Defined the Prüfer substitution (using path lifting) and derived ODE for  $\theta, r$ . The point however is that the boundary values for SL correspond to boundary values for  $\theta$ . Proved that there exists a unique solution for some short period of time.

# 2 Sturm-Liouville theory

$$\begin{aligned}\theta' &= (\lambda\rho - q) \sin^2 \theta + \frac{1}{p} \cos^2(\theta) = F(t, \theta), \\ r' &= \frac{1}{2} \left( \frac{1}{p} - (\lambda\rho - q) \right) r \sin(2\theta).\end{aligned}\tag{1}$$

$$r = r(a) \exp \left( \frac{1}{2} \int_a^t \left( \frac{1}{p(s)} - (\lambda\rho - q)(s) \right) \sin(2\theta(s)) ds \right).$$

Note that changing  $r(a)$  only scales  $u$  by a constant factor. Hence the zeroes of  $u$  can be located by studying  $\theta$ .

Since the right-hand-side is bounded,  $\theta'$  is bounded and hence  $\theta$  stays bounded in the maximal interval. Thus the maximal interval is  $[a, b]$ .

Zeroethly, note the following observations.

1. If  $\theta$  is a solution, then so is  $\theta + n\pi$  (with the same eigenvalue). Hence,  $\theta(a)$  can be assumed to be  $\geq 0$ . Moreover, by subtracting enough multiples of  $\pi$ ,  $\theta(a) \in [0, \pi)$ .
2. To find zeroes, simply find those values of  $t$  for which  $\theta(t) = n\pi$ .
3. Let  $n \geq 0$ . If there is a  $t_n$  so that  $\theta(t_n) = n\pi$ , then  $\theta'(t_n) > 0$  and hence  $\theta(t) > n\pi$  for  $t > t_n$  and sufficiently close to  $t_n$ . In fact, if there is a  $t > t_n$  such that  $\theta(t) = n\pi$ , then  $\theta'$  at that point is  $\leq 0$  which is a contradiction. Thus,  $\theta > 0$  on  $(a, b]$ .

The point is the following oscillation theorem.

**Theorem 1.** *Let  $\theta(t, \lambda)$  be a solution of the above ODE with  $\theta(a, \lambda) = \gamma \in [0, \pi)$ . Then  $\theta$  is continuous and it is strictly increasing in  $\lambda$ . Moreover,  $\lim_{\lambda \rightarrow \infty} \theta(t, \lambda) = \infty$  and  $\lim_{\lambda \rightarrow -\infty} \theta(t, \lambda) = 0$  for any  $t \in (a, b]$ .*

Given this theorem, let us prove the main theorem of SL: The boundary condition on  $b$  can be stated as  $\theta(b, \lambda) = \delta + n\pi$  for  $n = 0, 1, \dots$  provided  $\delta$  satisfies  $\beta_1 \sin(\delta) + \beta_2 \frac{\cos(\delta)}{p(b)} = 0$ . There is of course a unique  $\delta \in (0, \pi]$  satisfying it. For this value of  $\delta$ , by the theorem above and the intermediate value theorem, there is a unique  $\lambda$  such that  $\delta = \theta(b, \lambda)$ . Call this  $\lambda_0$ . (Note that  $u_0$  does not vanish on  $(a, b)$ .) Likewise, we can produce  $\lambda_1, \dots$  which form an increasing sequence. Since  $\delta_n \rightarrow \infty$ , so does  $\lambda_n$ . Why does the statement about zeroes follow?  $\square$

Now we shall prove the oscillation theorem. We first prove that  $\theta$  is strictly increasing

in  $\lambda$ . Note that  $\theta' = (\lambda\rho - q) \sin^2(\theta) + \frac{\cos^2(\theta)}{p}$ . Now since the right-hand-side is  $C^1$  in  $\lambda$ ,  $\theta(t, \lambda)$  is differentiable with respect to  $\lambda$  and the derivative is (jointly) continuous (and  $\theta(t, \lambda)$  is continuous jointly). We differentiate the equation.

$$\begin{aligned} \left(\frac{\partial\theta}{\partial\lambda}\right)' &= \rho \sin^2(\theta) + (\lambda\rho - q) \sin(2\theta) \frac{\partial\theta}{\partial\lambda} - \frac{\sin(2\theta)}{p} \frac{\partial\theta}{\partial\lambda} \\ &= \rho \sin^2(\theta) + \sin(2\theta) \frac{\partial\theta}{\partial\lambda} \left(\lambda\rho - q - \frac{1}{p}\right) \\ \frac{\partial\theta}{\partial\lambda}(a, \lambda) &= 0. \end{aligned} \tag{2}$$

Thus using an integrating factor, we can solve for  $\frac{\partial\theta}{\partial\lambda}$  and see that it is  $> 0$ . Hence  $\theta$  is strictly increasing in  $\lambda$ .

Now note that for  $\lambda$  large enough,  $\theta' \geq \frac{\lambda \sin^2(\theta) + 1}{C}$  (why?) This means that  $\theta$  is strictly increasing. If  $\lim_{\lambda \rightarrow \infty} \theta(t, \lambda) = L < \infty$  for some  $t = t_0$ , then the limit is  $\leq L$  for all  $t \in (a, t_0]$ .