1 Recap

1. Proved that there is a solution to the equation for θ on $[a, b]$.

2. Also proved that if the oscillation theorem

   Theorem 1. *Let $\theta(t, \lambda)$ be a solution of the above ODE with $\theta(a, \lambda) = \gamma \in [0, \pi)$. Then θ is continuous and it is strictly increasing in λ. Moreover, $\lim_{\lambda \to \infty} \theta(t, \lambda) = \infty$ and $\lim_{\lambda \to -\infty} \theta(t, \lambda) = 0$ for any $t \in (a, b]$.*

   is true, then the SL theorem can be proven.

3. Started the oscillation theorem. Were at this point: Note that for λ large enough, $\theta' \geq \frac{\lambda \sin^2(\theta) + 1}{C}$ (why?) This means that θ is strictly increasing. If $\lim_{\lambda \to \infty} \theta(t, \lambda) = L < \infty$ for some $t = t_0$, then the limit is $\leq L$ for all $t \in (a, t_0]$.

2 Sturm-Liouville theory

Continuing with the first limit's proof, note that $\int_\gamma^L \frac{d\theta}{\lambda \sin^2(\theta) + 1} \geq \frac{t_0 - a}{C}$. Now we notice that the LHS $\leq \int_0^{N\pi} \frac{d\theta}{\lambda \sin^2(\theta) + 1}$ for some N. Moreover, that is $\leq N \int_0^\pi \frac{d\theta}{\lambda \sin^2(\theta) + 1}$. Now take $u = \tan(\theta)$. Then the LHS $\leq 2N \int_0^\infty \frac{du}{1 + (1 + \lambda)u^2} = \frac{\pi N}{\sqrt{1 + \lambda}} \to 0$. This is a contradiction!

A small digression: We already know that regardless of λ, if $\theta(t_n) = n\pi$, then for all $t > t_n$, $\theta(t) > n\pi$. This means that the nth zero of u occurs when $t = n\pi$. Note that when $\lambda \to \infty$, if there is a sequence λ_k such $\lim_k t_n(\lambda_k) > a$, then again the above argument applies to arrive at a contradiction. This means that $t_n(\lambda) \to a$ when $\lambda \to \infty$.

Now we prove the second limit. We already know that $\theta > 0$ on $(a, b]$. Fix $t = t_1$. Suppose given an $\epsilon > 0$, we produce a λ such that $\theta(t_1, \lambda) < \epsilon$, we are done. To this end, (by shrinking ϵ if necessary), assume that there is a $\max(\gamma, \epsilon) < \gamma_1 \leq \pi - \epsilon$. Now consider a straight line $s(t)$ joining (a, γ_1) and (t_1, ϵ) (with negative slope m). Note that if for some λ, the graph of $\theta(t, \lambda)$ lies below $s(t)$, then $\theta(t_1, \lambda) < s(t_1) = \epsilon$ and we are done. It is easy to see that θ lies below the straight line for some $[a, a_1]$. Suppose for the sake of contradiction that there is a t so that $\theta(t, \lambda) > s(t)$. Then choosing the smallest such $t = t_*$, $\theta(t_*, \lambda) = s(t_*)$ and $\theta'(t_*, \lambda) \geq m$. Since $\theta(t_*) = s(t_*) = \gamma_1 + m(t_* - a)$, substituting for m and the upper bound for γ_1 we see that $\theta(t_*, \lambda) \in [\epsilon, \pi - \epsilon]$. Continued....