1 Recap

1. Finished the proof of the oscillation theorem for $\lambda \to \infty$ and almost for the other limit.

2 Sturm-Liouville theory

Recall that $s(t_*) = \theta(t_*)$ and $\theta'(t_*) \geq m$. Moreover, $\sin(\theta_*) \geq \sin(\epsilon)$. Thus for sufficiently negative λ, $\theta'(t_*, \lambda) < m$ and that is a contradiction. $\square$

We can in fact produce a good estimate for where the zeroes of u lie. To prove such an estimate, one proves a very important result (which is useful in Riemannian geometry too) called the Sturm comparison theorem for ODE of the form $(P_i u_i)' + Q_i u_i = 0$ for $i = 1, 2$ where $P_i > 0$ are C^1, and Q_i are continuous. If θ_i are the corresponding Prüfer phases, then $\theta_i' = Q_i \sin^2(\theta_i) + \frac{1}{P_i}\cos^2(\theta_i) = F_i(t, \theta_i)$.

Theorem 1. *Assume $P_1 \geq P_2 > 0$ and $Q_1 \leq Q_2$. Then between any two zeroes of a non-trivial u_1, there is at least one zero of every u_2 except if $u_2 = cu_1$. In the latter case, we have $P_1 = P_2, Q_1 = Q_2$ everywhere except possibly on the set $Q_1 = Q_2 = 0$.*

Proof. Suppose we have two zeroes $\theta_1(a) = n\pi$ and $\theta_1(b) = (n+1)\pi$ of u_1. Choose the initial data for θ_2 to be such that $(n+1)\pi > \theta_2(a) \geq \theta_1(a) = n\pi$. We claim that

$$\theta_1(t) \leq \theta_2(t) \; \forall \, t \in [a, b],$$

and that $\theta_1(b) = \theta_2(b)$ iff $\theta_1 \equiv \theta_2$.

Assume this claim. Then we see that $\theta_1(b) \leq \theta_2(b)$ and therefore unless $\theta_2(b) = (n+1)\pi$, θ_2 will cross $(n+1)\pi$ before b and hence u_2 will have a zero. If $\theta_2(b) = \theta_1(b)$, then $\theta_1(t) = \theta_2(t) = \theta$. Subtracting the two equations,

$$(Q_2 - Q_1)\sin^2(\theta) + \left(\frac{1}{P_2} - \frac{1}{P_1}\right)\cos^2(\theta) = 0.$$

When $\sin(\theta) = 0$, $\theta' > 0$ and hence the zeroes of u_i are isolated. Thus $Q_2 = Q_1$ everywhere and $P_2 = P_1$ unless $\cos(\theta) = 0$. On such intervals where $\cos(\theta) = 0$, θ is constant. Thus on such intervals, $Q_1 = Q_2 = 0$ because $\sin^2(\theta) = 1$. Using the explicit formula for r, we are done. We just have to prove the claim. To this end, we see that

$$\begin{aligned} (\theta_1 - \theta_2)' &\leq Q_2(\sin^2(\theta_1) - \sin^2(\theta_2)) + \frac{1}{P_2}(\cos^2(\theta_1) - \cos^2(\theta_2)) \\ (\theta_1 - \theta_2)(a) &\leq 0. \end{aligned} \tag{1}$$

Now suppose the claim is false. Then (why?) there exists an interval $[a_1, b_1]$ such that $(\theta_1 - \theta_2)(a_1) = 0$ and $(\theta_1 - \theta_2)(t) > 0$ on $[a_1, b_1]$. In that case, on this interval (why?),

$$\begin{aligned} (\theta_1 - \theta_2)' &\leq C|\theta_1 - \theta_2| = C(\theta_1 - \theta_2) \\ (\theta_1 - \theta_2)(a_1) &= 0 \end{aligned} \tag{2}$$

Thus by Gronwall, $\theta_1 \leq \theta_2$ which is a contradiction. Therefore, $\theta_1 \leq \theta_2$ on $[a,b]$. In fact, if $\theta_2(b) = \theta_1(b)$, then suppose $\theta_1(t) < \theta_2$ on a subinterval. Then the above argument shows that $\theta_1(b) < \theta_2(b)$ which is a contradiction and hence if equality holds, $\theta_1 \equiv \theta_2$. □