1 Recap

1. Location of zeroes of SL eigenfunctions.
2. Classification of 2×2 equilibria.
3. Definition of invariant subspaces.

2 Higher dimensions

In view of the Jordan canonical form, we define/recall generalised eigenvectors with generalised eigenvalue λ as vectors $v \neq 0$ such that $(A - \lambda I)^k v = 0$. The subspace of generalised eigenvectors corresponding to λ is called a generalised eigenspace. First we note that a generalised eigenspace is an invariant subspace. Indeed, bringing A to its Jordan canonical form, this follows easily.
By the theorem of the *real* Jordan canonical form, $\mathbb{R}^n$ is a direct sum of three kinds of subspaces: The ones with $Re(\lambda) < 0$: the stable subspace E^s, the unstable subspace E^u $Re(\lambda) > 0$, and the centre subspace E^c ($Re(\lambda) = 0$). These are all invariant subspaces. In fact, as $t \to \infty$, the flow takes E^s to 0 eventually and E^u to 0 as $t \to -\infty$. The analogous result for nonlinear systems is called the stable manifold theorem.

3 Nonlinear stability

We will largely study equations of the form $\vec{x}' = \vec{F}(\vec{x})$ where $\vec{x} \in \mathbb{R}^n$ (with the vector symbols usually omitted). This is an autonomous system of *dimension* n. An equilibrium point or a steady-state solution is a vector x_0 such that $F(x_0) = 0$. An equilibrium point can be isolated or non-isolated. If F is locally Lipschitz, by uniqueness, $x = x_0$ is the only solution with $x(t_0) = x_0$.
On paper, one can try to study things like $x' = x + t$ which are non-autonomous by reducing them to autonomous ones: Introduce a new variable $x_{n+1} = t$. Then $x_i' = F_i(x), x_{n+1}' = 1$. Unfortunately such a system *never* has equilibrium points even if the original one does (what is an example?). So this strategy does not always simplify matters.
Usually, assumptions are made on F so that the solution exists for all time for any or at least "most" initial data. Sometimes it is helpful to think of $\vec{x}(t)$ as a curve whose tangent vector is $\vec{F}(\vec{x})$. For this reason, $\vec{F}(\vec{x})$ is called a vector field and the solution is called an integral curve. Sometimes, $x_0 \to x(t)$ is called a time-t flow.
Def: The orbit $O(x_0)$ through x_0 is the solution of the ODE with $x(t_0) = x_0$. The positive orbit $O^+(x_0)$ is the solution with $t \geq t_0$. A periodic/closed orbit is a solution such that there exists a $T > 0$ such that for every t, $x(t + T) = x(t)$. The smallest such $T > 0$ is called the period. (What if the the infimum is 0?) Usually we exclude fixed points when we talk about periodic orbits. If a periodic orbit is isolated, then it is called a limit cycle.
Here are some properties about orbits.

Lemma 3.1. *Assume that $\vec{F}$ is locally Lipschitz.*

1. *Let $x(t)$ be an orbit passing through x_0. $x(t+c)$ is also a solution.*

2. *If $x(t_0) = y(t_1) = x_0$ and x, y are solutions, then $y(t) = x(t + t_0 - t_1)$.*

3. *Two orbits either coincide or do not intersect.*

4. *Suppose there exist $T > 0, t_0$ such that $x(t_0 + T) = x(t_0)$, then $x(t+T) = x(t)$ for all t.*

5. *There are no limit cycles for linear homogeneous autonomous systems $x' = Ax$. (For non-linear systems they can exist. Later on, we shall see that the Poincaré-Bendixon and Leinard's theorems give us sufficient conditions for $2D$ systems to have periodic solutions.)*

6. *Suppose x is a solution and $\lim_{t\to\infty} x(t) = \xi$ exists. Then ξ is an equilibrium point.*

Proof.

1. Easy (differentiate).

2. Again uniqueness and differentiation.

3. Uniqueness.

4. Uniqueness.

5. The explicit formula for x is $x(t) = e^{At}x_0$. Thus $x(t+T) = x(t)\,\forall\, t$ iff $(e^{AT}-1)x_0 = 0$. Either $x_0 = 0$ in which case it is an equilibrium point and does not count as periodic, or λx_0 also satisfies this condition for all λ and hence, for every periodic orbit, regardless of what neighbourhood we choose around it, there is another periodic orbit that intersects this neighbourhood.

6. Indeed, fix $h > 0$. Then $x(t+h) \to \xi$ as $t \to \infty$. By the mean-value theorem, $x_i(t+h) - x_i(t) = hf_i(x(\tilde{t}_i))$ where $\tilde{t}_i \in [t, t+h]$. As $t \to \infty$, so does $\tilde{t}_i$ and hence $f(\xi) = 0$.

□

Here are some examples.

1. $x' = x^2$: Only one equilibrium point. Unfortunately, the solution blows up in finite time.

2. $x' = \sin(x)$: Isolated equilibria. If we start at $x \in (2n\pi, (2n+1)\pi)$, then $\sin(x) > 0$ and hence x is increasing but $|x'| \leq 1$. The solution exists for time (positive and negative). It can never be $(2n+1)\pi$ (why?) Thus $\lim_{t\to\infty} x(t)$ exists. It must be $(2n+1)\pi$. Likewise for odd multiples.

3. Consider $r' = \sin(r)$ where $r(0) > 0$, $\theta' = 1$. This system has no equilibrium points. However, let $x = r\cos(\theta)$, $y = r\sin(\theta)$. Then there are limit cycles in the $x - y$ plane. Indeed, if we start with $r = n\pi$, r will remain $n\pi$ whereas θ changes. There can be no other periodic orbit (because r is monotonic) nearby.

4. If one wants to model an elastic pendulum, one uses the Duffing equations: $x' = y, y' = \pm x - x^3 - \delta y$. For the negative sign, the only equilibrium is the origin. For the positive sign one also has $(\pm 1, 0)$.

5. $x' = -y + x(x^2 + y^2)$ and $y' = x + y(x^2 + y^2)$ has only the origin as the equilibrium point. (Indeed, $y = x(x^2 + y^2)$, $x = -y(x^2 + y^2)$ and hence x, y have the same sign and opposite sign!)

6. $x'' + xx' = 0$: Writing as a first order system, $x' = v, v' = -xv$, we see that all points on the line $v = 0$ are equilibrium points. So no isolated ones.