1 Recap

1. Definitions of limit cycles, equilibria, etc.
2. Proved some properties.

2 Nonlinear stability

We still have to prove two parts of the lemma.

Lemma 2.1. *1. There are no limit cycles for linear homogeneous autonomous systems $x' = Ax$. (For non-linear systems they can exist. Later on, we shall see that the Poincaré-Bendixon and Leinard's theorems give us sufficient conditions for $2D$ systems to have periodic solutions.)*

2. Suppose x is a solution and $\lim_{t\to\infty} x(t) = \xi$ exists. Then ξ is an equilibrium point.

Proof. 1. The explicit formula for x is $x(t) = e^{At}x_0$. Thus $x(t+T) = x(t)\ \forall\ t$ iff $(e^{AT} - 1)x_0 = 0$. Either $x_0 = 0$ in which case it is an equilibrium point and does not count as periodic, or λx_0 also satisfies this condition for all λ and hence, for every periodic orbit, regardless of what neighbourhood we choose around it, there is another periodic orbit that intersects this neighbourhood.

2. Indeed, fix $h > 0$. Then $x(t+h) \to \xi$ as $t \to \infty$. By the mean-value theorem, $x_i(t+h) - x_i(t) = hf_i(x(\tilde{t}_i))$ where $\tilde{t}_i \in [t, t+h]$. As $t \to \infty$, so does $\tilde{t}_i$ and hence $f(\xi) = 0$.

□

Here are some examples.

1. $x' = x^2$: Only one equilibrium point. Unfortunately, the solution blows up in finite time.
2. $x' = \sin(x)$: Isolated equilibria. If we start at $x \in (2n\pi, (2n+1)\pi)$, then $\sin(x) > 0$ and hence x is increasing but $|x'| \leq 1$. The solution exists for time (positive and negative). It can never be $(2n+1)\pi$ (why?) Thus $\lim_{t\to\infty} x(t)$ exists. It must be $(2n+1)\pi$. Likewise for odd multiples.
3. Consider $r' = \sin(r)$ where $r(0) > 0$, $\theta' = 1$. This system has no equilibrium points. However, let $x = r\cos(\theta)$, $y = r\sin(\theta)$. Then there are limit cycles in the $x - y$ plane. Indeed, if we start with $r = n\pi$, r will remain $n\pi$ whereas θ changes. There can be no other periodic orbit (because r is monotonic) nearby.
4. If one wants to model an elastic pendulum, one uses the Duffing equations: $x' = y, y' = \pm x - x^3 - \delta y$. For the negative sign, the only equilibrium is the origin. For the positive sign one also has $(\pm 1, 0)$.
5. $x' = -y + x(x^2+y^2)$ and $y' = x + y(x^2+y^2)$ has only the origin as the equilibrium point. (Indeed, $y = x(x^2+y^2)$, $x = -y(x^2+y^2)$ and hence x, y have the same sign and opposite sign!)

6. $x'' + xx' = 0$: Writing as a first order system, $x' = v, v' = -xv$, we see that all points on the line $v = 0$ are equilibrium points. So no isolated ones.

3 Liapunov stability

We confine ourselves mainly to Liapunov stability: An isolated equilibrium point $\bar{x}$ of $x' = f(x)$ is said to be Liapunov stable if given $\epsilon > 0$, there is a $\delta > 0$ such that for any solution with $\|x(t_0) - \bar{x}\| < \delta$, $\|x(t) - \bar{x}\| < \epsilon$ for all $t > t_0$. Otherwise it is said to be Liapunov unstable.
Asymptotic stability: An isolated equilibrium $\bar{x}$ is asymptotically stable if it is Liapunov stable and if $\|x(t_0) - \bar{x}\| < b$, then $\lim_{t\to\infty} \|x(t) - \bar{x}\| = 0$.
In the case of linear equations, $x' = Ax$, if $\det(A) \neq 0$, then 0 is the only equilibrium. If the eigenvalues have negative real parts, then it is asymptotically stable. If even one of them has a positive real part, it is unstable. If they have ≤ 0 real parts (with at least one zero real part), then it *can be* stable or can be unstable too (but certainly not asymptotically so - why? Be careful in that eigenvectors can be complex!).