1 Recap

1. Examples of equilibria and limit cycles.

2. Liapunov stability -An isolated equilibrium point $\bar{x}$ of $x' = f(x)$ is said to be Liapunov stable if given $\epsilon > 0$, there is a $\delta > 0$ such that for any solution with $\|x(t_0) - \bar{x}\| < \delta$, $\|x(t) - \bar{x}\| < \epsilon$ for all $t > t_0$,
   and asymptotic stability - An isolated equilibrium $\bar{x}$ is asymptotically stable if it is Liapunov stable and if $\|x(t_0) - \bar{x}\| < b$, then $\lim_{t\to\infty} \|x(t) - \bar{x}\| = 0$.

2 Liapunov stability

One important idea is to approximate a nonlinear system by its Newton/first-order approximation. This is called linearisation. $f(\bar{x}+y) = f(\bar{x})+Df(\bar{x})y+O(\|y\|^2)$ (assume f is smooth for simplicity). We can now consider the linear system $y' = Ay$ and try to look at its stability properties. They may not always reflect that of the nonlinear system but sometimes they do. Here is an example of a theorem:

Theorem 1. *Suppose the eigenvalues of $A = Df(\bar{x})$ (where $\bar{x}$ is an equilibrium) all have negative real parts and f is C^2. Then the equilibrium point is asymptotically stable.*

This theorem is a special case of Perron's theorem:

Theorem 2. *Let A be a real $n \times n$ matrix whose eigenvalues all have negative real parts. Consider $x' = Ax + f(t, x)$ where f is continuous, locally Lipschitz in x, defined on $\mathbb{R} \times \mathbb{R}^n$ and satisfies: Given $\epsilon > 0$, there exists a $\delta > 0$ so that $\|f(t, x)\| < \epsilon\|x\|$ for all $\|x\|\delta$ (δ is independent of t). Then any solution x with sufficiently small $x(0)$ exists for all $t \geq 0$ and $x(t)$ tends to 0 as $t \to \infty$.*

Note that even existence for all $t \geq 0$ is not straightforward! For instance, if $x' = x^2$, then $A = 0$ and the theorem above does not apply. (Indeed, in this case the solution blows up in finite time.) On the other hand, if $x' = -\mu x + x^2$ where $\mu > 0$, then the theorem does apply. But even more simply, $x' \geq -\mu x$ and hence x is bounded below (at least for finite positive time). Now if $0 < x(0) \leq \frac{\mu}{2}$, then $x' \leq -\frac{\mu}{2}x$ for small enough t. For all such t, $x \leq x_0 e^{-\mu/2t} \leq \frac{\mu}{2}$. Thus x stays in this interval for all $t \geq 0$ and indeed converges to 0 as $t \to \infty$. This is the main idea of the proof.

Proof. Firstly note that $\|e^{tA}\| \leq Ke^{-\sigma t}$ for some $K > 1, \sigma > 0$ as we saw in the HW. Now by Picard, there exists a unique solution with any given initial data x_0 for on a maximal interval $[0, h)$. Moreover, we can prove (how) that $x(t) = e^{At}x_0 + \int_0^t e^{(t-s)A} f(s, x(s))ds$. By the hypothesis on f, there exists a $\delta > 0$ such that $\|f(t, x)\| \leq \frac{\sigma}{2K}\|x\|$ for all $\|x\| \leq \delta$. Now assume that $\|x_0\| < \frac{\delta}{K}$. We claim that for all such x_0, the solution exists for all time and satisfies $\|x\| \leq \delta$. Consider the set of all t for which $\|x\| \leq \delta$. This set obviously contains $[0, t_*]$ for some $t_* > 0$. Consider the maximum such t_*. For all $t \leq t_*$, $\|x(t)\| \leq Ke^{-\sigma t}\|x_0\| + \frac{\sigma}{2} \int_0^t e^{-\sigma(t-s)}\|x(s)\|ds$. Multiplying by $e^{\sigma t}$ and using Gronwall, $e^{\sigma t}\|x\| \leq K\|x_0\|e^{\sigma/2t}$. Hence, $\|x\| \leq K\|x_0\|e^{-\sigma/2t} < \delta$ and hence the solution extends beyond t_* and satisfies $\|x\| \leq \delta$ beyond it as well. Thus t_* is not finite. $\square$

Before we proceed, here is an important definition: An equilibrium $\bar{x}$ is called hyperbolic if all the eigenvalues of $Df(\bar{x})$ have non-zero real parts.