1 Recap

1. Proof of Perron's theorem (the hypotheses needed to be modified - f was locally Lipschitz in x to ensure uniqueness). Also proved a corollary (for which f can be assumed to be C^2).

2 Liapunov stability

Here are some more examples:

1. $x' = -y + x(x^2 + y^2)$, $y' = x + y(x^2 + y^2)$. Here A has eigenvalues $\pm\sqrt{-1}$. So the theorem above does not apply. The linearisation at the origin has the origin as a stable (but not asymptotically stable) equilibirum point. However, considering $r = \sqrt{x^2 + y^2}$, $r' = r^3$ and hence if $r_0 > 0$, then r blows up in finite time. Thus the origin is unstable. On the other hand, if we consider $x' = -y - x(x^2 + y^2)$, $y' = x - y(x^2 + y^2)$, then $r' = -r^3$ and hence the origin is asymptotically stable.

2. The Duffing system with negative sign: $x' = y, y' = x - x^3 - \delta y$. The equilibria are $(0,0), (\pm 1, 0)$. The linearisation at $(0,0)$ has eigenvalues $\frac{-\delta \pm \sqrt{\delta^2 + 4}}{2}$. This point is linearly unstable when $\delta \geq 0$. At $(\pm 1, 0)$, the eigenvalues are $\frac{-\delta \pm \sqrt{\delta^2 - 8}}{2}$. Thus for $\delta > 0$, the eigenvalues have negative real parts and by the theorem above, these equilibria are asymptotically stable. If $\delta = 0$, the points are linearly stable but not linearly asymptotically stable. It is hard to analyse the nonlinear system.

3. $x' = y, y' = x - x^3$. We claim that this system isnot stable at the origin. Indeed if it were, there exists $(0 < x_0 < 1, y_0 > 0)$ such that the solution exists on $[0, \infty)$ and satisfies $\|(x, y)\| < 1/2$. But in this regime, $y' \geq 0$ and hence $x' \geq y_0$ and hencce for large t we have a contradiction.

4. The van der Pol system: $x' = y, y' = \mu(x^2 - 1)y - x$. The origin is the only equilibrium. The linearisation has eigenvalues $\frac{-\mu \pm \sqrt{\mu^2 - 4}}{2}$. If $\mu > 0$ it is linearly (and hence nonlinearly) asymptotically stable. If $\mu < 0$ it is linearly unstable. When $\mu = 0$ we have a linear system which is stable.

5. $x' = y, y' = -x + 2y - 2x^2$. Subtracting, $(x - y)' = (x - y) + 2x^2$ and hence $x - y = (x - y)_0 e^t + e^t \int_0^t 2x^2 e^{-s} ds > (x_0 - y_0)e^t$. Hence if $x_0 - y_0 > 0$, $x - y$ runs off to infinity. Thus this point is not stable.